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SPATIAL ARRANGEMENTS IN ARCHITECTURE AND
MECHANICAL ENGINEERING - SOME ASPECTS OF
THEIR REPRESENTATION AND CONSTRUCTION

Christopher Francis Earl B.A., M.Sc.

Submitted for the degree of Doctor of Philosophy
in the Design Discipline, Faculty of Technology,
Open University

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ABSTRACT

Spatial arrangements in architecture and mechanical engineering are represented by incidence structures and classified according to properties of these incidence structures. The relationships between classes are given by ornamentation operations and the construction of elements in fundamental classes by substructure replacement operations. Thus representations of the spatial arrangements for possible designs are generated.

Planar maps represent spatial arrangements in architectural plans. The edges correspond to walls and vertices to incidence between walls. Plans represented by 3-vertex connected maps are ornamented by rooting and extension operations. Further ornamentation specifies access between regions. Plans with all regions adjacent to the exterior correspond to outerplane maps. Trivalent maps represent an important class of plans. Fundamental plans with r internal regions and s regions adjacent to the exterior are represented by $[r,s]$ triangulations. Ornamentations of simple $[r,s]$ triangulations are specified which represent plans with rectangular regions. Plans with walls aligned along two directions are represented by rectangular shapes whose maximal lines correspond to contiguous aligned walls. Rules of construction for various classes are given and the incidence structures of maximal lines and regions are characterized.

Spatial arrangements in machines are represented by systems whose blocks correspond to links and vertices to joints. The dual systems are also used. Coplanar kinematic chains with revolute pairs are classified according to mobility and connectedness.

Two fundamental classes are considered. First, the chains with binary joints, represented by simple graphs and constructed by two new methods: (i) suspended chain and cycle addition and (ii) subgraph replacement. Second, the chains with binary links which are constructed by subgraph replacement.

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Published work

Some preliminary work for Chapters 2 and 3 was done in collaboration with Lionel March and is reported in two jointly authored papers:

(i) "On counting architectural plans", *Environment and Planning B* 4 (1977), 57-80 and (ii) "Architectural applications of graph theory" in *Applications of Graph Theory* edited R.J. Wilson and L.W. Beineke, Academic Press, London, pp. 327-255.

Most of the material contained in Chapters 4 and 5 has been published in two papers by the author: (i) "Rectangular shapes", *Environment and Planning B* 7 (forthcoming) and (ii) "The representation of kinematic chains", *Environment and Planning B* 6 (1979) 455-468.

The above papers make no mention that the research reported had any connection with an Open University higher degree.

Availability

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SPATIAL ARRANGEMENTS AND INCIDENCE STRUCTURES

1.1 Spatial arrangements

Herbert Simon in 'The Sciences of the Artificial' (Simon 1969) remarks that "Since much of design, particularly architectural and engineering design, is concerned with objects or arrangements in real Euclidean two-dimensional or three-dimensional space, the representation of space and things in space will necessarily be a central topic in the science of design". This thesis is concerned with the spatial arrangement of elements in architectural and mechanical systems.

There are four steps in the examination of spatial arrangements.

- (1) To identify the kinds of elements and provide ways of representing the relationships between the elements. This step attends to properties which characterise the spatial arrangements.
- (2) To provide means of constructing the representations of spatial arrangements which possess certain properties.
- (3) To produce descriptions of representations.
- (4) To combine and transform representations. This step might introduce further properties for the spatial arrangements.

The four steps are considered as representation, generation, description and ornamentation. Representation and description come under the broad heading representation, and generation and ornamentation, under construction.

The spatial arrangement of an architectural plan consists in the relations among the elements comprising the plan. The arrangement depends upon the elements considered, the level of detail at which they are specified, and the relations among them. Spatial arrangements in

architectural plans have broadly two kinds of basic element. First there are the elements corresponding to lines, that is, walls or partitions and second the regions, that is rooms or activity areas. Associated with these basic elements are three kinds of relationship. First incidence of lines, second incidence of regions and lines and third adjacency of regions. The adjacency between regions in a plan may correspond to access between the regions. This type of adjacency provides an access relation between regions. Higher order adjacency relations between regions may also be considered in terms of the number of regions which separate two regions. This might be called the distance relation between regions. If architectural plans are considered whose walls lie on grid lines then there is a further relation between the lines, namely the alignment of walls. This relation imposes the order of the grid on the walls.

Elements need not be basic lines or regions. Aggregates of lines and regions might compose elements among which relations are defined. In addition there are multiple types of lines and regions in an architectural plan and consequently many relations.

Three dimensional spatial arrangements can also be considered. The basic elements are planes and regions. The relations which compose spatial arrangements are essentially incidence of planes, incidence of regions and planes, and adjacency between regions. Similarly to the two-dimensional case there can be further relations, including the access, distance and alignment relations.

The spatial arrangement in a mechanical design consists in the relations among the elements composing the machine. The basic elements in a machine are the links or rigid bodies and the relations are the

connections between links to form kinematic pairs and the grouping of kinematic pairs to form joints. This thesis is concerned only with these elements and relations and does not address more complex geometric aspects of spatial arrangements or the kinematic properties of machines. For spatial arrangements in mechanical design there is no way of avoiding their essentially three dimensional character. There are, in general, no convenient "plans" which may be considered.

The representation of the spatial arrangements in two-dimensional architectural plans can be given in terms of graphs which represent the incidence structures involved. Graphs embedded on the plane or sphere are used because the extra structure due to the embedding allows the relative disposition of the elements to be considered. The representation of spatial arrangements on grids is usually in terms of line segments on the grid since the order or alignment relation of the lines is conveniently represented by the grid itself. However, the alignment relation may be incorporated into a graph type representation with appropriate labellings. The representations in three dimensions are similar, but the graphs are not necessarily planar, although they are considered to be embedded in three-dimensional euclidean space.

The representation of the spatial arrangements in mechanical design is in terms of graphs and systems. The incidence of links in kinematic pairs may be represented by a graph. The incidence of links at joints, that is, collections of kinematic pairs can be represented conveniently in terms of systems in which either the joints compose the links, or dually the links compose the joints.

For architectural and mechanical designs the representations of spatial arrangement provide ways of describing the various incidence

patterns between the elements composing the arrangements. In an examination of spatial arrangement, attention is focussed on classes of incidence pattern which have particular interest in architectural or mechanical design. There are two tasks to perform at this stage. First, to provide means of generating all possible incidence patterns in the classes of interest. Thus given a set of characterising properties to provide means of generating all incidence patterns with the properties. The second task is to examine the characterising properties of the classes of incidence patterns and deduce other properties which incidence patterns in particular classes will possess. The two tasks are interdependent. The second may be facilitated by the first which in some sense provides the "local structure" of the class. Conversely, the first task may require, for elegant or efficient generation, the results of the second, which provides the "global structure" of the class. The results can be used in three ways.

- (1) To produce extensive catalogues of possibilities from which selection can be made in design.
- (2) To produce constructive procedures whereby incidence patterns can be generated in response to particular design requirements.
- (3) To indicate subclasses of interest for which characterising properties and means of generation may be established.

In some cases it is possible to enumerate the incidence patterns in various classes. This can be useful in several ways. First, the production of an enumeration exposes the structure of a class. An enumeration might consist of a recursive formula or a generating function equation. Second, the result of an enumeration gives the sizes of sets from which selection is made in design. Third, the comparison of the sizes of classes and subclasses suggests where it

might be desirable to consider certain classes as combinations and transformations of smaller more manageable subclasses.

The means of generation examined in this thesis depend upon replacing a sub-pattern in an incidence pattern by another to produce a new incidence pattern. For representation by graphs, this takes the form of a 'subgraph replacement' operation. For the representation of spatial arrangements on grids this may take the form of a 'sub-shape replacement' operation.

The incidence patterns constructed in this thesis will usually consist of basic classes from which other patterns are obtained by ornamentation operations. These are generally of two types.

- (1) Combination of patterns, usually by identifying sub-patterns in two patterns.
- (2) Transformation of patterns by (i) transforming the incidence patterns but preserving the types of elements and relations and (ii) preserving the incidence patterns but transforming the types of elements and relations.

1.2 Graphs; terminology and definitions

A graph $G = (V, E)$ is an ordered pair of sets; the vertices V and edges E which consist of unordered pairs of elements of V . There is an isomorphism between graphs $G = (V, E)$ and $G_1 = (V_1, E_1)$ if there are bijections $\phi : V \rightarrow V_1$ and $\psi : E \rightarrow E_1$ such that $\phi(e) = \psi(e)$ for all $e \in E$.

A subgraph of $G = (V, E)$ induced by $V' \subseteq V$ is the graph $G[V'] = (V', E')$ where $E' = \{e : e \in E \cap \mathcal{P}_2(V')\}$. Also let $G[V - V'] = G - V'$. A subgraph of $G = (V, E)$ induced by $E' \subseteq E$ is the graph $G[E'] = (V', E')$ where $V' = \{v : v \in e, e \in E'\}$. A subgraph of $G = (V, E)$ obtained by removing $E' \subseteq E$ from G is $G - E' = (V, E - E')$.

A path in a graph $G = (V, E)$ is an ordered set of vertices and distinct edges $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$, such that $(v_i, v_{i+1}) = e_i$. A subgraph $G[V']$ of $G = (V, E)$ is connected if each pair of vertices belongs to a common path. A maximal connected subgraph $G[V']$ of $G = (V, E)$ is a component of G . A graph is connected if it has exactly one component.

Consider a connected graph $G = (V, E)$. A k-vertex cut in G is $V' \subset V$, $|V'| = k$, such that $G - V'$ is not connected. A k-edge cut in G is $E' \subseteq E$, $|E'| = k$, such that $G - E'$ is not connected. A vertex of degree k corresponds trivially to a k -edge cut. A non-trivial k-edge cut in G is $E' \subseteq E$, $|E'| = k$, such that $G[E - E']$ is not connected. G is k-vertex connected if there is no k' -vertex cut with $k' < k$, and k-edge connected if there is no k' -edge cut with $k' < k$. It is non-trivially k-edge connected if there is no non-trivial k' -edge cut with $k' < k$.

A path with $v_1 = v_{k+1}$ is a k-cycle and a graph without 1 or 2-cycles is simple. A connected graph without cycles is a tree.

1.3 Planar Maps

A topological graph in a 2-dimensional oriented surface S has edges and vertices which are disjoint subsets of S and each edge is an open arc in S whose endpoints are its incident vertices. If the topological graph is finite, connected and has at least one edge then it is called a map. Each component of the complement of a map in S is homeomorphic to an open disc and is called a face of the map. A map M has cells $C(M)$ consisting of the vertices $V(M)$, edges $E(M)$ and faces $F(M)$. Two cells of different type are incident if one is contained in the boundary of the other. Two vertices or two faces are

adjacent if they are incident to a common edge.

Let the ordered set of edges and vertices incident to a face $f \in F(M)$ be denoted by $B(f)$, the boundary of the face. Similarly let the ordered set of edges and faces incident to a vertex $v \in V(M)$ be denoted by $B(v)$. Let $i(e,f)$ and $i(v,f)$ denote the number of times $e \in E(M)$ and $v \in V(M)$ respectively, occur in $B(f)$ and define the degree or valency of f as $\sum i(e,f)$ where the sum is taken over all $e \in E(M)$. Also let $i(v,e)$ denote the number of times e occurs in $B(v)$ and define the degree or valency of a vertex as $\sum i(v,e)$ where the sum is taken over all $e \in E(M)$. Note that $i(v,e) \leq 2$ and $i(e,f) \leq 2$. If $i(e,f) = 2$ then e is an isthmus and if $i(v,e) = 2$ then e is a loop. Finally, a map in the oriented 2-sphere is called a planar map.

A map M in an oriented surface can be represented by a permutation pair (Tutte 1979). Each edge in $E(M)$ corresponds to two opposite directed edges or darts. Let θ be a permutation on the set of darts D that takes each dart into its opposite. The darts directed away from vertices occur in cyclic orders which correspond to the cycles of a permutation P on D . The cyclic orders of darts which have a given face of M on their right, say, are cycles of the permutation $P\theta$ on D . Since the graph underlying the map is connected, P and θ generate a group of permutations which acts transitively on D . The permutation pair (P,θ) acting on D represents the map M . Conversely, a permutation pair (P,θ) acting on a set D such that θ has all cycles of length two and (P,θ) generate a group of permutations which acts transitively on D , represents a map M in which $V(M)$ corresponds to the cycles of P , $E(M)$ to the cycles of θ and $F(M)$ to the cycles of $P\theta$. The permutation pair $(P\theta,\theta)$ acting on D represents another map M^* , the dual of M . The map M is on a

surface with Euler characteristic $|V(M)| - |E(M)| + |F(M)|$, and M is planar if the Euler characteristic is two. Attention is confined to planar maps.

An orientation preserving isomorphism of a planar map $M = (P, \theta)$ acting on D and a map $M_1 = (P_1, \theta_1)$ acting on D_1 is a bijection $\phi : D \rightarrow D_1$ such that for each $d \in D$, $\phi(P(d)) = P_1(\phi(d))$ and $\phi(\theta(d)) = \theta_1(\phi(d))$. This combinatorial definition is equivalent to a topological definition (Walsh 1971). There is an orientation preserving isomorphism between two planar maps M and M_1 if and only if there is an orientation preserving homeomorphism ψ of the 2-sphere onto itself such that $\psi(V(M)) = V(M_1)$, $\psi(E(M)) = E(M_1)$ and $\psi(F(M)) = F(M_1)$. Denote this by $\psi(M) = M_1$ and say that there is an orientation preserving homeomorphism between M and M_1 . A homeomorphism between two planar maps is defined similarly except that orientation may not be preserved.

A planar map M with a distinguished face $f \in F(M)$ is called a plane map $M(f)$. Under a stereographic projection of M which takes f onto an unbounded region in the plane, a representation of M in the plane is obtained with face f corresponding to the external region. An orientation preserving boundary homeomorphism of a plane map $M(f)$ onto another plane map $M_1(f_1)$ is an orientation preserving homeomorphism ψ of the 2-sphere onto itself such that $\psi(M) = M_1$ and $\psi(f) = f_1$. A boundary homeomorphism between two plane maps is defined similarly except that orientation may not be preserved.

A plane map $M(f)$ is rooted by distinguishing an $e \in B(f)$ together with a direction on e . If e is an isthmus the direction may be chosen in either sense and if e is not an isthmus then the direction is chosen in the sense of the orientation in f . An orientation

preserving boundary homeomorphism ψ of plane maps is a root homeomorphism of rooted plane maps if ψ preserves the root edge and its direction. If the negative end of root edge $e \in E(M)$ is vertex $v \in V(M)$ then denote a rooted plane map by $M(f, e, v,)$ and call f and v the root face and root vertex respectively. A root homeomorphism of a rooted plane map onto itself maps each vertex, edge and face onto itself (Tutte 1963). Thus a rooted map has no symmetry.

For a class of planar maps a convenient combinatorial definition of equivalence may be given in place of the topological definitions above. A planar map M is non-singular if all edges and faces in $B(v)$, $v \in V(M)$ are distinct. There is an orientation preserving homeomorphism of non-singular planar maps M and M_1 if and only if there is a bijection $\phi : C(M) \rightarrow C(M_1)$ such that $\phi(V(M)) = V(M_1)$, $\phi(E(M)) = E(M_1)$, $\phi(F(M)) = F(M_1)$, ϕ and ϕ^{-1} preserve incidence and orientation. Denote this by $\phi(M) = M_1$ and call ϕ an orientation preserving isomorphism of non-singular planar maps. Isomorphism which may not preserve orientation is defined similarly. Boundary and root isomorphisms are also defined for non-singular plane and rooted plane maps respectively.

Let $\phi : M(f, e, v) \rightarrow M_1(f_1, e_1, v_1)$ be an orientation preserving root isomorphism between non-singular rooted plane maps. The action of ϕ on f , e and v determines ϕ completely (Brown 1963).

A triangular map is a non-singular planar map in which each face is incident with exactly three distinct edges. Triangular maps are used to represent the incidences in planar maps, between vertices and edges, edges and faces, and faces and vertices. A derivable map M' (Tutte 1963) is a triangular map for which there is an ordered

partition $W(V), W(F), W(E)$ of $V(M')$ such that each $e \in E(M')$ has vertices in distinct classes and each vertex in $W(E)$ is degree four. A planar map M can be constructed from a derivable map M' by setting up a bijection between $W(V), W(F), W(E)$ and the vertices, faces and edges of M which preserves ordering. The map M' represents the incidences in M and is called the derived map of M . If the construction of M has $W(V)$ and $W(F)$ interchanged then the map M^* dual to M is derived from M' . There is an orientation preserving homeomorphism of two planar maps if and only if there is an orientation preserving isomorphism of the derived maps.

Let $M(f,e,v)$ denote a rooted plane map. There are corresponding rootings of M' . Let $a \in W(V)$ correspond to v , $b \in W(E)$ correspond to e and $c \in W(F)$ correspond to f . Distinguish the face in $F(M')$ with ordered vertices (a,b,c) by the orientation. If edge (a,b) is distinguished, a $(W(V), W(E))$ rooting of M' is defined. Similarly if edges (b,c) or (c,a) are distinguished, $(W(E), W(F))$ and $(W(F), W(V))$ rootings, respectively, are defined (figure 1.1(ii)). The $(W(E), W(F))$ rooting induces a rooting on the dual map M^* . There is a 1-1 correspondence between rooted plane maps under root homeomorphism and each of the sets of $(W(V), W(E))$, $(W(E), W(F))$ and $(W(F), W(V))$ rooted derivable maps. If (v, v_1) are the endpoints of e in M and $a_1 \in W(V)$ corresponds to v_1 then $(W(E), W(V))$, $(W(F), W(E))$ and $(W(V), W(F))$ rootings are defined by considering the face in $F(M')$ with ordered vertices (a_1, c, b) by the orientation (figure 1.1 (iii)). The $(W(F), W(E))$ rooting induces a rooting on the dual map M^* . This induced rooting on the dual is used throughout the thesis (figure 1.1 (iv)).

Finally, in this section on the definition of planar maps, the

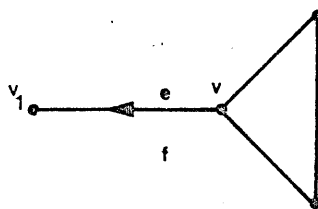
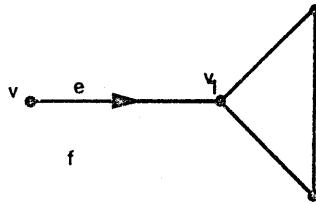
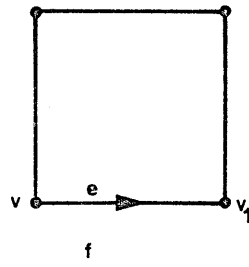
notions of connectedness are introduced. A 1-vertex separator in a map M , not a single loop or single edge, is $(v; f)$ $v \in V(M)$, $f \in F(M)$ such that $i(v, f) \geq 2$. If M has no 1-vertex separator then M is 2-vertex connected. Note that a planar map M , $|e(M)| \geq 2$ is non-singular if and only if M is 2-vertex connected and that the dual map M^* is 2-vertex connected if and only if M is 2-vertex connected. A 2-vertex separator in a map M is $(v_1, v_2; f_1, f_2)$ $v_1 \neq v_2 \in V(M)$, $f_1, f_2 \in F(M)$ such that $i(f_1, v_1) = i(f_2, v_1) = i(f_1, v_2) = i(f_2, v_2) = 1$. Note that 1 and 2-vertex cuts in the underlying graph of a map M do not necessarily correspond to 1 and 2-vertex separators respectively. A 2-vertex separator induces a partition of $E(M)$ into two non-empty sets E_1, E_2 called a 2-vertex separation. It is proper if $|E_i| \geq 2$ $i=1, 2$. If M is a 2-vertex connected map with no proper 2-vertex separation and has more than one edge then M is 3-vertex connected. The dual M^* is 3-vertex connected if and only if M is 3-vertex connected.

A 1-edge separator in a map M is $(e; f)$ $e \in E(M)$, $f \in F(M)$ such that $i(f, e) = 2$. M is 2-edge connected if there is no 1-edge separator. Note that a 1-edge separator corresponds to an isthmus, which in turn corresponds to a loop in the dual map. A loop in a 2-edge connected map corresponds to an isthmus in the dual map, thus duality does not preserve 2-edge connectedness. A 2-edge separator in a map M is $(e_1, e_2; f_1, f_2)$, $e_1 \neq e_2 \in E(M)$ and $f_1, f_2 \in F(M)$ such that $i(f_1, e_1) = i(f_1, e_2) = i(f_2, e_1) = i(f_2, e_2) = 1$. If M is a 2-edge connected map with no 2-edge separator then M is 3-edge connected.

A 1-vertex separator in a map M corresponds in the derived map M' to two distinct edges with common endpoints in $W(V)$ and $W(F)$, that is a multiple edge. A 1-edge separator corresponds to a multiple

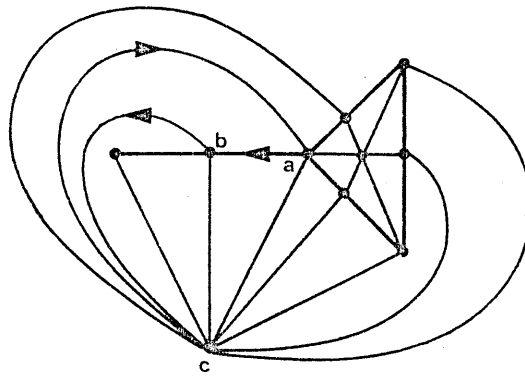
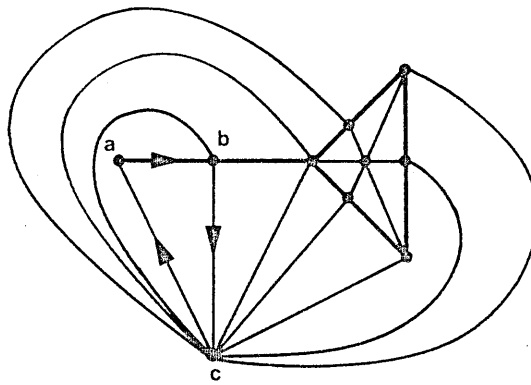
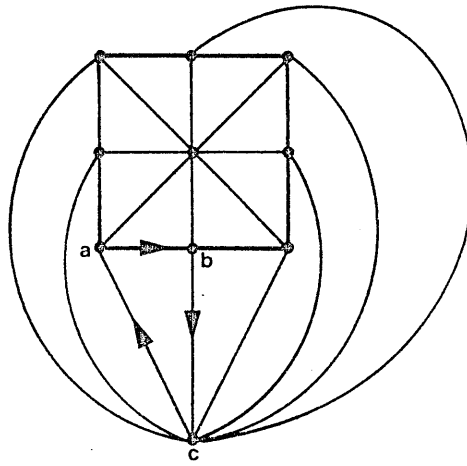
edge with endpoints in $W(E)$ and $W(F)$. A 1-edge separator in a map M corresponds to a loop in M^* , and a 2-edge separator to a multiple edge in M^* .

In the next chapter the planar maps are examined in more detail using a classification by vertex connectedness. The edges of a planar map represent the walls in an architectural plan and the vertices represent the incidence of walls. The vertex connectedness of a planar map gives a global property to this incidence relation between walls. The edge connectedness does not relate directly to the incidence between walls, although it might be used as an architecturally relevant means of classification. Note, however, that vertex and edge connectedness are related in the sense that a map is 2 or 3-edge connected if it is 2 or 3-vertex connected, respectively.



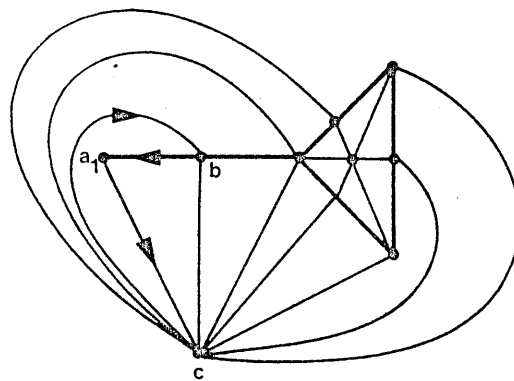
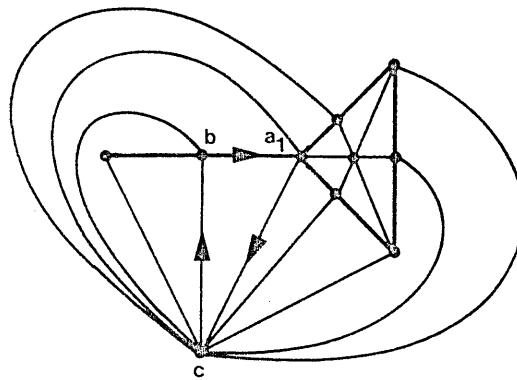
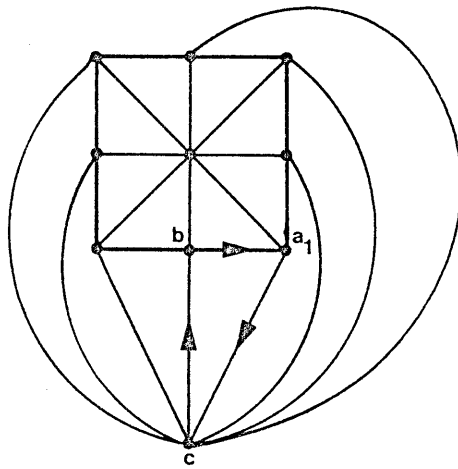
(i)

FIGURE 1.1 Rootings of derived and dual maps



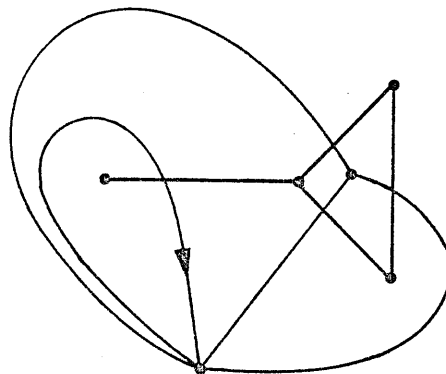
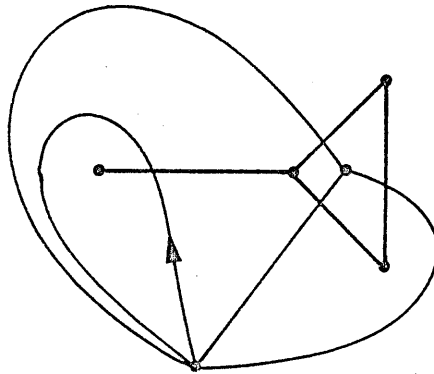
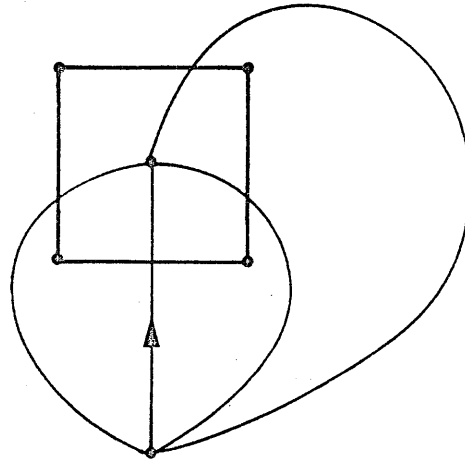
(ii)

FIGURE 1.1



(iii)

FIGURE 1.1



(iv)

FIGURE 1.1

CHAPTER 2

PLANAR MAPS AND ARCHITECTURAL PLANS

2.1 Introduction

Planar maps comprise a representation of arrangements of lines or walls in architectural plans, in which edges represent walls and the vertices represent incidence or connection between walls. In an architectural plan the walls have many types. Perhaps they are movable partitions, internal walls, load bearing walls or external walls containing windows and doors. Representation by a planar map assumes that all walls are identical. However, if a plane map is used then the external walls are distinguished. A rooted plane map distinguishes a particular external wall as the "facade", say.

The faces in a planar map representing the arrangement of lines or walls correspond to regions in an architectural plan, and the edges correspond to boundaries of regions. The duals of planar maps represent the adjacencies among regions in a plan. The regions may also be of many types. Perhaps, some are subdivided according to a decomposition of activities, some are circulation spaces and others living areas. There may be external spaces and courtyards. Representation by a planar map assumes that all regions are the same type. However, if a plane map is used, one region is distinguished as external. A rooted plane map also distinguishes another region incident to the distinguished edge.

The derived map of a planar map represents region-wall, wall-vertex and region-vertex incidences. In fact the derived map may be seen as a convenient representation of all the incidences and adjacencies in an architectural plan. The purpose of this chapter is to examine

classes of planar, plane and rooted plane maps which represent spatial arrangements in architectural plans.

2.2. Classification by connectedness

This section examines the classification of planar maps by connectedness, and the properties of the corresponding architectural plans.

A topological graph embedded in the oriented 2-sphere which is not necessarily connected and which may have isolated vertices is called a pseudo map. The faces of a pseudo map may be multiply connected. If a set of edges $E' \subseteq E(M)$ are removed from a pseudo map M the result is a pseudo map $M - E'$. If the isolated vertices are removed from $M - E'$ the pseudo map $M[E - E']$ is obtained. A component of a pseudo map M is either an isolated vertex or $M[E - E']$, $E' \subseteq E(M)$, such that $M[E - E']$ is a map and there is no $E'' \subset E'$ such that $M[E - E'']$ is a map. A pseudo map may be considered as the embedded "sum" of its components and a disconnected architectural plan is represented by the sum of its components.

A 1-vertex separator $(v;f)$ in a planar map indicates that the corresponding architectural plan comprises two or more parts joined at a single vertex. A 1-vertex separator $(v;f)$ in a plane map $M(f)$ indicates an "external" joining and a 1-vertex separator $(v;g)$, $g \neq f$ indicates an "internal" joining of plans.

A 2-vertex separator $(v,w; g,h)$ in a plane map $M(f)$ represents a plan comprising two parts with an "external" joining at two vertices if $f \in \{g,h\}$ and "internal" otherwise. A 2-edge separator $(d,e; g,h)$ in a plane map $M(f)$ represents a "through" room or corridor if

$f \in \{g, h\}$ and if $f \notin \{g, h\}$ the 2-edge separator represents a pair of doubly incident regions forming a ring.

In order to consider the 1,2 and 3-vertex connected rooted plane maps two additional operations are considered. First, 1-vertex extension and second, 1-edge extension.

2.2.1 Extensions

Let $M(f, e, v)$ and $M_1(f_1, e_1, v_1)$ denote two rooted plane maps (figure 2.1). Let $w \in V(M)$ be incident to $g \in F(M)$ and consider consecutive edges, $b, d \in B(w)$ at w . Add a loop c in g with vertex w , creating a new face h . Also add a loop c_1 to M_1 in face f_1 , with vertex v_1 . Both c and c_1 have directions defined by the orientations in g and f_1 , respectively. Identify w and v_1 , c and c_1 together with their directions, and also identify the interiors of f_1 and h . Finally remove $c = c_1$ and merge the faces h_1 and g . The new rooted plane map with distinguished face f , root edge e and root vertex v is a 1-vertex extension of $M(f, e, v)$ by $M_1(f_1, e_1, v_1)$ at $w \in V(M)$. This operation can be expressed in terms of the elements in the derived maps (Tutte 1963).

Let $M(f, e, v)$ and $M_1(f_1, e_1, v_1)$ be 2-vertex connected rooted plane maps (figure 2.2). Let $c \in E(M)$, $c \neq e$ and $g, h \in F(M)$ incident to c . Let c be directed from w to u , $w, u \in V(M)$, by the orientation in g . Let e_1 be directed from v_1 to w_1 . Split the edge c by an edge d in h to create a new face n . Also split e_1 by an edge d_1 in h_1 with endpoints v_1 and w_1 . Identify the exterior of n_1 with the interior of n , u with v_1 , w with w_1 , c with e_1 and d with d_1 . Finally remove $c = e_1$ and $d = d_1$, merging the face g with f_1 and h with h_1 . The new rooted plane map with distinguished face f , root-edge e and root-vertex v is the 1-edge extension of $M(f, e, v)$ by $M_1(f_1, e_1, v_1)$ at $c \in E(M)$.

A particular case of 1-edge extension occurs when $M_1 (f_1, e_1, v_1)$ is the cycle map, that is with all vertices degree two. This 1-edge extension subdivides an edge in M by inserting degree two vertices. A 1-edge extension by an n -cycle, $n > 2$, is equivalent to a sequence of $n-2$ 1-edge extensions by 3-cycles.

Consider a 1-edge extension by a 3-cycle followed by a 1-vertex extension at the new vertex. This is called a 1-augmented 1-vertex extension (figure 2.3(i)). Now consider a 1-edge extension by a cycle followed by a 1-edge extension on one of the new edges. Two cases are important.

- (1) A 1-edge extension by a 3-cycle, which subdivides an edge (u, w) with vertex s , followed by a 1-edge extension on (u, s) or (s, w) is called a 1-augmented 1-edge extension (figure 2.3(ii)).
- (2) A 1-edge extension by a 4-cycle, which subdivides an edge (u, w) with vertices s_1 and s_2 , followed by a 1-edge extension on (s_1, s_2) is called a 2-augmented 1-edge extension (figure 2.3(iii)).

Note that the operations of 1-vertex and 1-edge extension are formulated as applying to rooted plane maps. They may be formulated for planar maps but any constructive application would require knowledge of the symmetry of the map used in the extension. This information is provided implicitly by the set of rooted versions of a given planar map.

2.2.2. Construction and enumeration

A rooted plane map can be constructed by a sequence of 1-vertex extensions of a 2-vertex connected rooted plane map M where each extension is at a distinct vertex of M ; there are $|V(M)|$ "sites" for extension. Given a rooted plane map the 1-vertex extensions used to

construct it are uniquely determined. This result is used to relate the generating function for B_n , the number of 2-vertex connected rooted plane maps up to root homeomorphism with n edges and the generating function for A_n , the number of rooted plane maps with n edges. The A_n can be found by enumerating the $(W(V), W(E))$ rooted derivable maps (Tutte 1963) to be

$$A_n = \frac{2(2n)!3^n}{n!(n+2)!}, \quad (1)$$

and consequently the B_n are shown (Tutte 1963) to be

$$B_n = \frac{2(3n-3)!}{n!(2n-1)!}. \quad (2)$$

A 2-vertex connected rooted plane map, not a loop or single edge, can be constructed by a sequence of 1-edge extensions of a 3-vertex connected rooted plane map M at non-root edges of M . Consider the following classes of 2-vertex connected rooted plane maps.

- (1) Those derived by 1-edge extensions from the 3-vertex connected rooted plane map in figure 2.4(i)
- (2) Those derived by 1-edge extensions from the 3-vertex connected rooted plane map in figure 2.4(ii)
- (3) Those derived by 1-edge extensions from a 3-vertex connected map with more than three edges.

These three classes of 2-vertex connected rooted plane maps are mutually disjoint. The generating functions for the maps in classes (1) and (2) are given in terms of the generating function for B_n . Given a 2-vertex connected rooted plane map in class (3) the 1-edge extensions used to construct it are uniquely determined. The generating function for the maps in class (3) may thus be expressed in terms of the generating functions for B_n and C_n , the number of 3-vertex

connected rooted plane maps with n edges. The generating functions for B_n and C_n are thus related and the C_n can be calculated (Tutte 1963). Table 2.1 and figure 2.5 give the 1, 2 and 3-vertex connected rooted plane maps for small numbers of edges.

The absence of loops or isthmuses provides a further classification of planar maps. A rooted plane map without loops can be constructed from a 2-vertex connected rooted plane map without loops by a sequence of 1-vertex extensions using rooted plane maps without loops, at distinct vertices of the original map. A relation between the generating functions for loopless rooted plane maps and 2-vertex connected rooted plane maps can thus be established. Since the latter is known the former may be derived. Similar considerations apply to the isthmusless and isthmusless, loopless rooted plane maps.

Consider the general application of 1-vertex extension and 1-edge extension for the construction of rooted plane maps. Any rooted plane map can be constructed by a sequence of 1-vertex extensions, using 2-vertex connected rooted plane maps, from a 2-vertex connected rooted plane map. Note that the loop map and the edge map are 2-vertex connected. If the loop (edge) map is not used in 1-vertex extensions using 2-vertex connected rooted plane maps then loopless (isthmusless) maps are constructed. If neither loop nor edge maps are used then loopless, isthmusless maps are constructed. Any 2-vertex connected rooted plane map, not a loop or single edge can be constructed from a 3-vertex connected rooted plane map by a sequence of 1-edge extensions by 3-vertex connected rooted plane maps. Note that a 3-cycle is 3-vertex connected and thus edges may be subdivided.

The enumerations for the 1, 2 and 3-vertex connected rooted plane maps

suggest that the full set of planar maps is too large, even with a small number of edges, to be useful in itself. It is perhaps more conveniently considered as derived from the 2-vertex connected planar maps by 1-vertex extensions, which in turn are considered as derived from the 3-vertex connected planar maps by 1-edge extension. The 3-vertex connected planar maps represent a basic set of possibilities for the incidence patterns of walls in architectural plans in the sense that the incidence patterns may be constructed from these basic units.

2.2.3 2-vertex connected planar maps

The 2-vertex connected planar maps are now examined in more detail. Further classification is considered together with the sets of plane maps up to orientation preserving boundary homeomorphisms.

The 2-vertex connected rooted plane maps can be classified according to the total number of edges and the number of edges in the boundary of the distinguished face (Brown 1963) (table 2.2 and figure 2.6). The corresponding 2-vertex connected plane maps up to orientation preserving homeomorphisms have also been examined (table 2.3) together with the types of rotational symmetry encountered in these maps. The corresponding architectural plans are classified according to the number of external walls, internal walls and their "potential" rotational symmetry.

The 2-vertex connected rooted plane maps can also be classified according to both the number of edges, faces, and edges in the boundary of the distinguished face. Brown and Tutte (1964) give a generating function equation for the numbers of such maps and solve it to give the number of 2-vertex connected rooted plane maps with n edges and

m faces as

$$\frac{(2n-m-3)! \cdot (n+m-3)!}{(n-m-1)! \cdot (m-1)! \cdot (2n-2m-3)! \cdot (2m-3)!} \quad (3)$$

2.2.4 3-vertex connected planar maps

There are well established results for the generation of 3-vertex connected planar maps. The first is due to Tutte (1961) and states that a 3-vertex connected planar map, whose underlying graph is not a wheel can be constructed from a 3-vertex connected planar map M with fewer edges by one of the following operations (figure 2.7):

- (1) Splitting a vertex of M incident with four or more edges and adding a new edge incident with the resulting new vertices.
- (2) Joining two vertices in the boundary of a face, not already joined by an edge, with an edge lying entirely inside the face.

Both operations add an edge, the first preserves the number of faces, whilst the second increases the number of faces by one.

The second result states that a 3-vertex connected planar map may be constructed from the tetrahedron map with six edges by a sequence of the following operations (figure 2.8):

- (1) Joining two vertices in the boundary of a face, not already joined by an edge, with an edge lying entirely inside the face.
- (2) Subdividing an edge with a vertex and using (1) with the new vertex.
- (3) Subdividing two edges in the boundary of a face and using (1) with both new vertices.

The classification of 3-vertex connected rooted plane maps according to the number of edges and faces has been investigated (Mullin and Schellenberg 1968). The number of 3-vertex connected rooted plane maps with n edges and m faces is obtained (table 2.4 and

and figure 2.9).

The 3-vertex connected planar maps are often referred to as polyhedra or 3-polytopes on account of Steinitz's Theorem (Grünbaum 1967) on the realizability of planar maps as 3-polytopes. The number of distinct polyhedra up to homeomorphism (enantiomorphs are not distinguished) with n edges and m faces has been found by construction for $m \leq 8$. (Table 2.5 and figure 2.9.) A catalogue of the dual planar maps with up to eight vertices is given in Britton and Dunitz (1973).

The 3-vertex connected planar maps form the basic set of possibilities for the spatial arrangement of walls in an architectural plan. However, as noted previously it is often convenient to use the set of rooted versions when manipulating and combining these incidence patterns.

2.2.5 Dual planar maps

If a planar map represents the spatial arrangement of lines or walls in an architectural plan then the dual planar map corresponds to the spatial arrangement of regions under the adjacency relation. These planar maps are classified by connectedness in exactly the same way as for planar maps representing lines or walls. Note that a 1-vertex extension of a rooted plane map corresponds to a 1-vertex extension of the dual rooted plane map and similarly for 1-edge extension. Further classification will be in terms of the numbers of vertices and edges rather than faces and edges, since the vertices now represent regions in architectural plans.

Some simple relationships between planar maps and their duals are mentioned, in addition to the preservation of connectedness referred

to in section 1.3. A loop in a planar map M corresponds to an isthmus in the dual M^* , and an isthmus in M corresponds to a loop in M^* . A degree two vertex in M corresponds to a digon face in M^* and a 2-edge cut in M corresponds to a pair of edges with common endpoints.

2.3 The access relation

The access relation between regions in an architectural plan often generates a plane tree that is, a planar map with exactly one face, in which vertices represent regions and edges the access between regions. The plane trees can be rooted by distinguishing the vertex corresponding to the external space as root vertex. If the root vertex is degree one then there is internal access between any pair of interior regions and if the root vertex is degree greater than one then it is necessary to "go outside" in order to gain access between parts of the plan.

2.3.1 Tree rooted plane maps

Suppose that M is a rooted plane map with n edges and m vertices. Let T be a distinguished spanning tree of M . Then M together with the distinguished spanning tree forms a tree-rooted map $M(T)$. The cotree of T in $M(T)$ has $n - m + 1$ edges. Subdivide each edge of this cotree by two special vertices and remove the edges between the special vertices. A tree is obtained with $2(n - m + 1)$ special vertices all degree one, and is rooted by a suitable convention. The rooted plane map M can be described by this rooted plane tree and the spanning tree of M^* corresponding to the cotree of T in M , again suitably rooted (Mullin 1966). Thus tree rooted maps are conveniently described by pairs of rooted plane trees. A tree-rooted map represents the region adjacencies in an architectural plan together with a specification of

which adjacencies are realized by physical access.

2.3.2 Enumeration of rooted plane trees

A plane tree is a planar map with precisely one face and a plane tree is rooted by distinguishing a vertex. A general enumerative result for rooted plane trees is that the number of rooted plane trees with n edges, $v(i)$ non-root vertices of degree i , and root vertex degree k is (Mullin, 1966)

$$\frac{k (n-1)!}{\prod_{i=1}^{\infty} v(i)!} \quad (4)$$

if $n = \sum_{i=1}^{\infty} v(i)$ and $2n = \sum_{i=1}^{\infty} i v_i + k$ and zero otherwise.

The number of rooted plane trees with i ordinary non-root vertices and $2j$ special non-root vertices of degree one, is given by Mullin (1966) as

$$\frac{(2i+2j)!}{i!(i+1)!2j!} \quad (5)$$

As a special case the number of rooted plane trees with $i+1$ vertices is

$$\frac{2i!}{i!(i+1)!} \quad (6)$$

Since, tree-rooted plane maps are described by pairs of rooted plane trees, they may be enumerated. Also, since rooted plane trees with a given vertex partition are enumerated, the tree rooted plane maps with n edges, m vertices and $v(i)$ non-root vertices of degree i and root vertex of degree k , can be enumerated (Mullin, 1966) in the following way.

The number of rooted plane trees with $v(i)$ non-root vertices, $i \geq 2$, $v(1) + 2j$ non-root vertices degree one, and root vertex degree k is given by formula (4) as

$$n(i, j, k) = \frac{k \left(\sum_{i=1}^{\infty} v(i) + 2j - 1 \right)!}{(v(1) + 2j)! \prod_{i=2}^{\infty} v(i)!} \quad (7)$$

if $\sum_{i=1}^{\infty} (i-2) v(i) = 2j - k$ and zero otherwise.

Thus the number of rooted plane trees with $v(i)$ ordinary non-root vertices degree $i \geq 1$, $2j$ special non-root vertex degree one and root-vertex degree k is

$$\binom{v(1) + 2j}{2j} n(i, j, k) = \frac{k \left(\sum_{i=1}^{\infty} v(i) + 2j - 1 \right)!}{(2j)! \prod_{i=1}^{\infty} v(i)!} \quad (8)$$

if $\sum_{i=1}^{\infty} (i-2) v(i) = 2j - k$ and zero otherwise.

2.3.3 Enumeration of tree rooted plane maps

A tree rooted map $M(T)$ with n edges and m vertices is described by a pair of rooted plane trees. First, the spanning tree in M^* corresponding to the cotree of T in $M(T)$. This rooted plane tree has $n - m + 1$ edges. Second, the tree obtained from T by "adding" special vertices. This rooted plane tree has $2(n - m + 1)$ special vertices. Thus the number of tree rooted maps with n edges, m vertices, $v(i)$ non-root vertices degree $i \geq 1$, and root vertex degree k is given by formulae (6) and (8)

$$\frac{(2(n - m + 1))!}{(n - m + 1)! (n - m + 2)!} \cdot \frac{k \left(\sum_{i=1}^{\infty} v(i) + 2(n - m + 1) - 1 \right)!}{(2(n - m + 1))! \prod_{i=1}^{\infty} (v(i)!)} \quad .$$

$$= \frac{k (2n - m)!}{(n - m + 2)! (n - m + 1)! \prod_{i=1}^{\infty} (v(i)!)} \quad (9)$$

if $\sum_{i=1}^{\infty} (i - 2) v(i) = 2(n - m + 1) - k$ and zero otherwise.

2.4 Outerplane maps

An outerplanar map has a face which shares a boundary edge with every other face. An outerplane map is a plane map with distinguished face adjacent to every other face. A rooted outerplane map is a rooted plane map with distinguished face adjacent to every other face. A map with one face or a plane tree is outerplanar.

2.4.1 Construction of outerplane maps

A rooted outerplane map may be constructed from a 2-vertex connected rooted outerplane map by a sequence of 1-vertex extensions by rooted outerplane maps at distinct vertices in the original map. The 1-vertex extensions by plane trees may occur at any vertex and in any face. However, 1-vertex extensions by rooted outerplane maps which are not plane trees must occur at a vertex in the boundary of the distinguished face and must lie in the distinguished face.

Analogous to the general case of rooted plane maps, a rooted outerplane map may be constructed from a 2-vertex connected rooted outerplane map by a sequence of 1-vertex extensions by 2-vertex connected rooted outerplane maps such that each extension which is not an edge map lies in the distinguished face of the previous rooted outerplane map.

A 2-vertex connected rooted outerplane map may be constructed from a 3-vertex connected rooted outerplane map by a sequence 1 or 2-augmented

1-edge extensions by 2-vertex connected rooted outerplane maps at distinct edges of the original map such that $f = g$ in figure 2.2.

The sequence of 1 and 2-augmented 1-edge extensions is not unique for the construction of a given 2-vertex connected rooted outerplane map whereas the use of ordinary 1-edge extensions would give a unique sequence. However, by using 1 or 2-augmented 1-edge extensions it is easier to ensure that outerplanarity is preserved.

Consider the general application of 1 and 2-augmented 1-edge extensions in the construction of 2-vertex connected rooted outerplane maps. A 2-vertex connected rooted outerplane map may be constructed from a 3-vertex connected rooted outerplane map by a sequence of 1 and 2-augmented 1-edge extensions by 3-vertex connected rooted outerplane maps such that $f = g$ in figure 2.2.

The outerplane maps are particularly relevant in the architectural context since it is often required to give all (or at least most) of the internal spaces an external aspect, perhaps for daylighting or external access (Lynes 1977).

2.4.2 Weak dual plane map

The weak dual plane map of a plane map $M(f)$ is the map obtained from M^* by deleting the vertex $v^* \in V(M^*)$ corresponding to f and its incident edges. The union of v^* , its incident edges and incident faces in $M(f)$, is the distinguished face of the weak dual plane map. The weak duals represent the adjacencies among the internal regions in an architectural plan. The weak dual plane maps of an outerplane map, has all its vertices in the boundary of the distinguished face. Every such map is the weak dual plane map of an outerplane map. Generally it will be the weak dual of many outerplane maps.

A plane map $M(f)$ in which the boundary of f , $B(f)$, is a simple closed curve and each vertex in $V(M)$ lies in $B(f)$, is a general dissection of a polygon. Dissections of a polygon are 'tree-like' plane maps in the following sense. Let D be a rooted dissection of a polygon and D^* its corresponding rooted dual plane map. Suppose $v^* \in V(D^*)$ corresponds to the distinguished face in D . Subdivide each edge incident to v^* with vertices v_1, \dots, v_k where v_1 lies on the root edge of M^* . Remove the segments (v_i, v^*) $i = 1, \dots, k$ and the vertex v^* . The result, with v_1 as the root vertex, is a planted plane tree, that is, a rooted plane tree in which the root vertex is degree one.

The number of rooted dissections of a polygon, in which the exterior face has degree m and there are $v(i)$ internal faces of degree i is given, on using the formula for rooted plane trees with a given partition, by

$$\binom{m + v(1) - 1}{v(1)} \frac{(m + \sum_{i=1}^{\infty} v(i) - 1)!}{(m + v(1) - 1)! \prod_{i=2}^{\infty} (v(i))!} = \frac{(m + \sum_{i=1}^{\infty} v(i) - 1)!}{(m - 1)! \prod_{i=1}^{\infty} (v(i))!} \quad (10)$$

if $\sum (i - 2) v(i) = m - 2$ and zero otherwise.

The general dissections of a polygon might possess loops, giving internal faces of degree one. A dissection of a polygon (not the loop map) is 2-vertex connected if and only if it has no loops. A 2-vertex connected dissection of a polygon may have internal faces of degree two. The 2-vertex connected dissections of a polygon in which each face has degree greater than two have been extensively investigated (Read 1978, Harary Palmer and Read 1975). These dissections are precisely the weak dual plane maps of the 3-vertex connected outerplane maps.

The weak dual plane maps of outerplane maps may be constructed from an edge map or a 2-vertex connected dissection of a polygon by a sequence of 1-vertex extensions by edge maps and 2-vertex connected dissections such that each 1-vertex extension which is not a loop lies in the distinguished face of the previous map.

2.5 Conclusion

The representation of architectural plans by planar and plane maps has been the subject of previous investigations, which have generally concentrated on the arrangement of regions. The first approaches used a graph in which vertices represented regions and edges, the required adjacencies between regions (Levin 1964, Cousin 1970, Steadman 1976). The possible embeddings of the adjacency graphs were examined, although not in great detail. This shortcoming was rectified by Grason (1970) who used a "planar graph grammar" to construct possible embeddings and to supplement, in a general way, a set of required adjacencies. However, in the examples considered by Grason, the adjacency graph or map was required to satisfy at all stages of construction, certain "well-formedness" conditions corresponding to the realization of the adjacency graph in an architectural plan with rectangular boundary and all regions rectangular. These plans and their adjacency graphs are investigated in section 3.4. In this chapter the set of possible incidence structures among walls and regions in architectural plans which a "planar graph grammar" can generate, have been examined.

The treatment differs fundamentally from the previous approaches not only in the fact that sets of possible incidence

structures are examined, but also in that architectural plans are considered, in the first instance, to be spatial arrangements of walls. The incidence structures of the walls are represented by planar or plane maps. The spatial arrangements of regions are considered to be derived. Given an incidence structure of walls the regions can be considered as the faces in the corresponding planar or plane map, as indicated in this chapter. However, if the walls have further properties, such as doors, windows or moveable partitions then the architecturally relevant regions might well require a different interpretation.

The development of this chapter is now placed in the context of the scheme given in section 1.1. This serves as a conclusion to the chapter and an introduction to the next. Section 2.1 identifies the kinds of elements and the ways of representing the relationships between the elements. Section 2.2 provides the means of constructing representations of the spatial arrangements corresponding to architectural plans with different degrees of connectedness. In this section, some descriptions of the representations are given together with enumeration results.

The 3-vertex connected planar maps are considered as a basic set for representing the incidence structures in architectural plans. Specifying a distinguished face and rooting an edge may be considered as ornamentations. The addition or extension operations allow further rooted plane maps to be constructed from the "components" in the basic set.

The relative disposition of an original map and a map used

to extend it are crucial in considering an addition of maps. The rooting of maps, since it 'destroys' symmetry, allows not only a specification of where to apply the extension (relative to the root in the original map), but also how to apply it (relative to the root in the extension map). The extension operations can be considered as ornamentations in which two incidence structures are combined.

In section 2.3 the region adjacency structure of an architectural plan is ornamented by adding the access relation to the properties under consideration in the representation. The specification of an access structure is considered as an ornamentation which transforms the region adjacency structure by specifying additional properties of the relations, (adjacencies) between elements (regions). The addition of an access structure can also be considered as an ornamentation of the incidence structure of walls. Additional properties of elements (walls) are specified whilst the relations between the elements remain unchanged. This emphasises the fact that walls are considered as elements in the wall incidence structures and as relations in the region adjacency structures.

The particular case in which the access structure is a tree is investigated briefly. Further work in this area might include an examination of distance measures on these spanning trees. The mean distance and depth of the trees could be relevant properties in the architectural context. Other types of access structure could also be examined; for instance, those which consist of a single cycle with short spurs (Doyle and Graver 1976). Notions

of connectedness could be developed for access structures in architectural plans. The relation between access structures and the incidence structures of walls and regions which accommodate them, could offer fruitful ground for investigation.

In section 2.4 a particular subclass of planar maps, namely the outerplane maps, is considered for attention in the architectural context. Characterizing properties and means of construction and ornamentation are established. Further work might investigate those planar maps in which the regions are all adjacent to a small subset of regions. This subset of regions might correspond to the major spaces in a plan, such as halls, corridors, the external spaces and courtyards.

The scheme of investigation set out in section 1.1 has been demonstrated in many respects. However, there has been limited attention to the classes of incidence structure of architectural relevance. The next chapter considers such classes and appropriate ornamentation operations. It concludes with an extensive discussion of the ornamentation of a certain class of planar maps which provides architectural plans with rectangular regions.

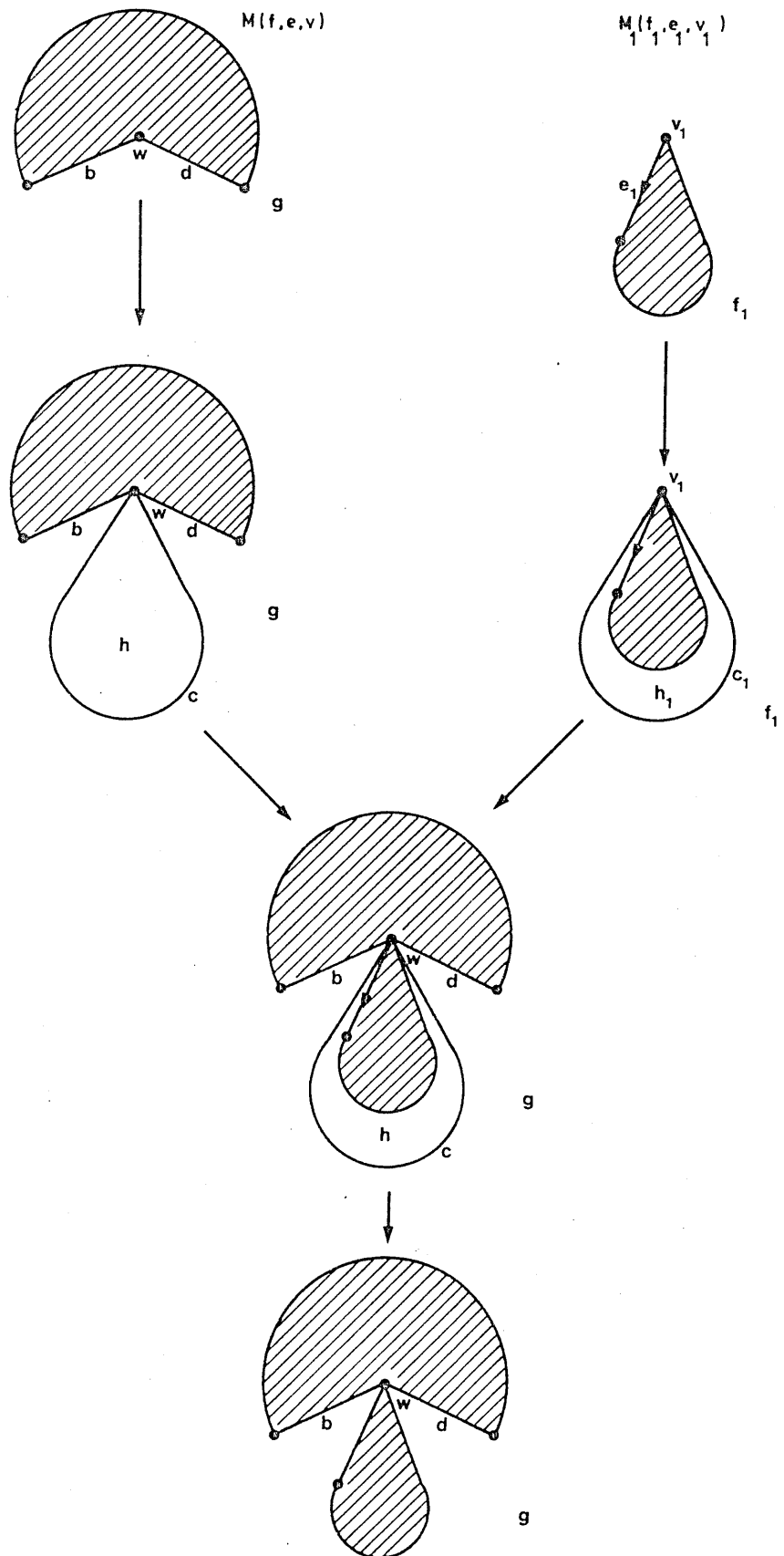


FIGURE 2.1 1-vertex extension

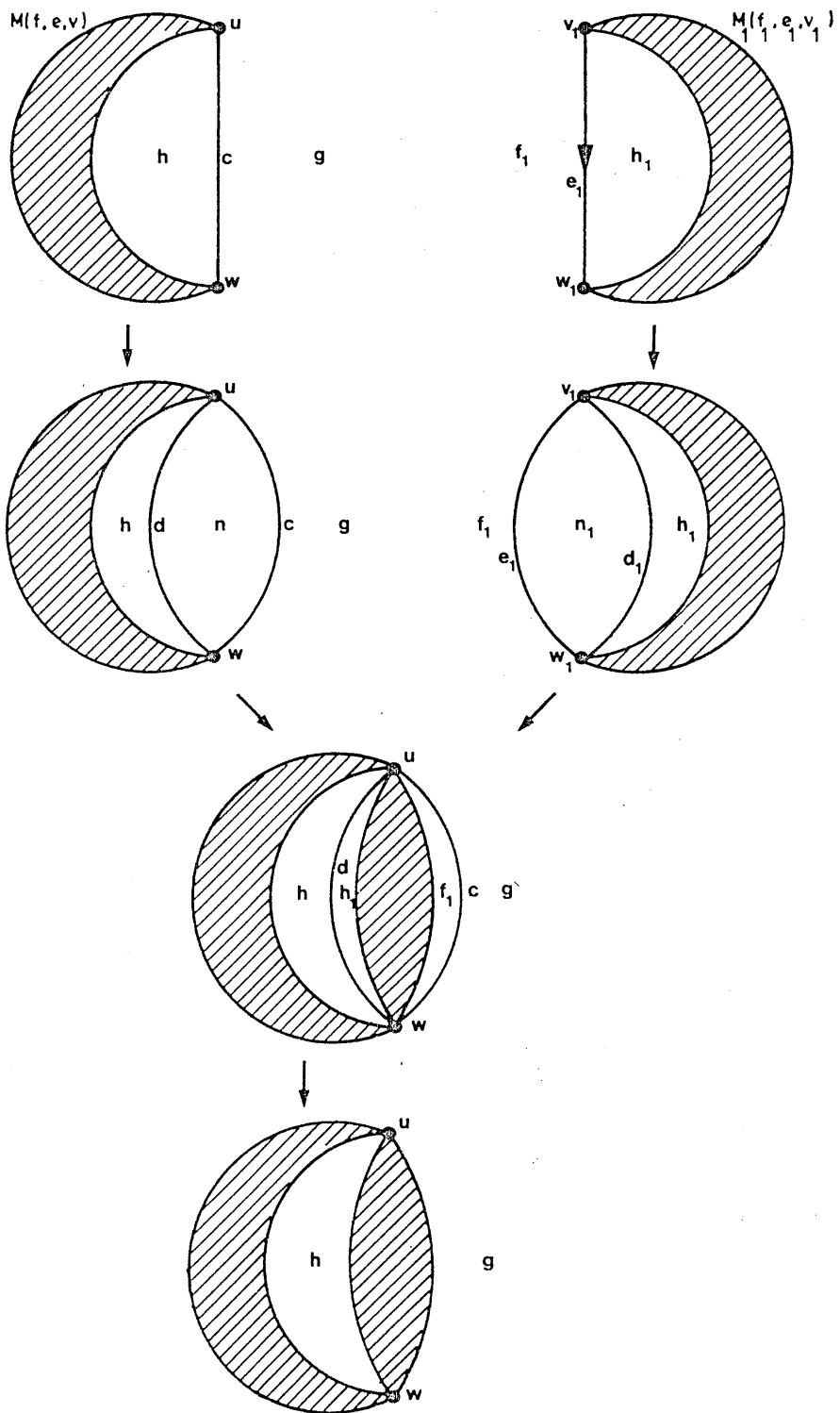
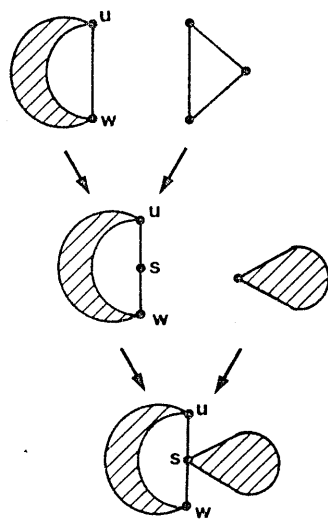
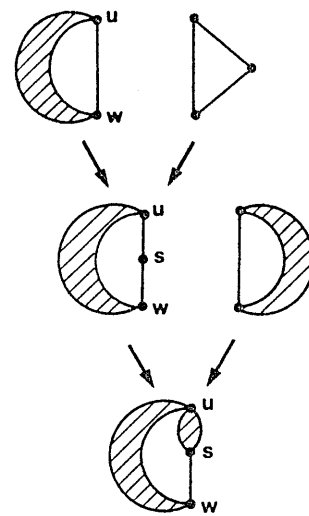


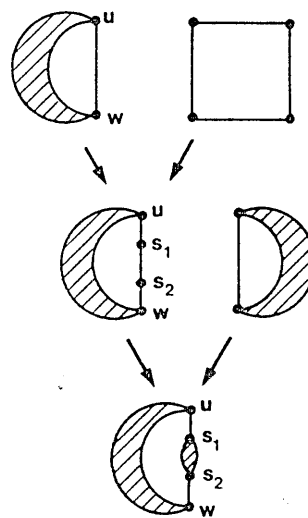
FIGURE 2.2 1-edge extension



(i)

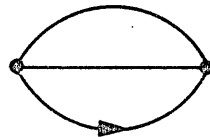


(ii)

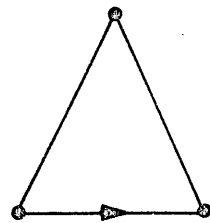


(iii)

FIGURE 2.3 Augmented extensions



(i)



(ii)

FIGURE 2.4 3-vertex connected rooted plane maps with three edges

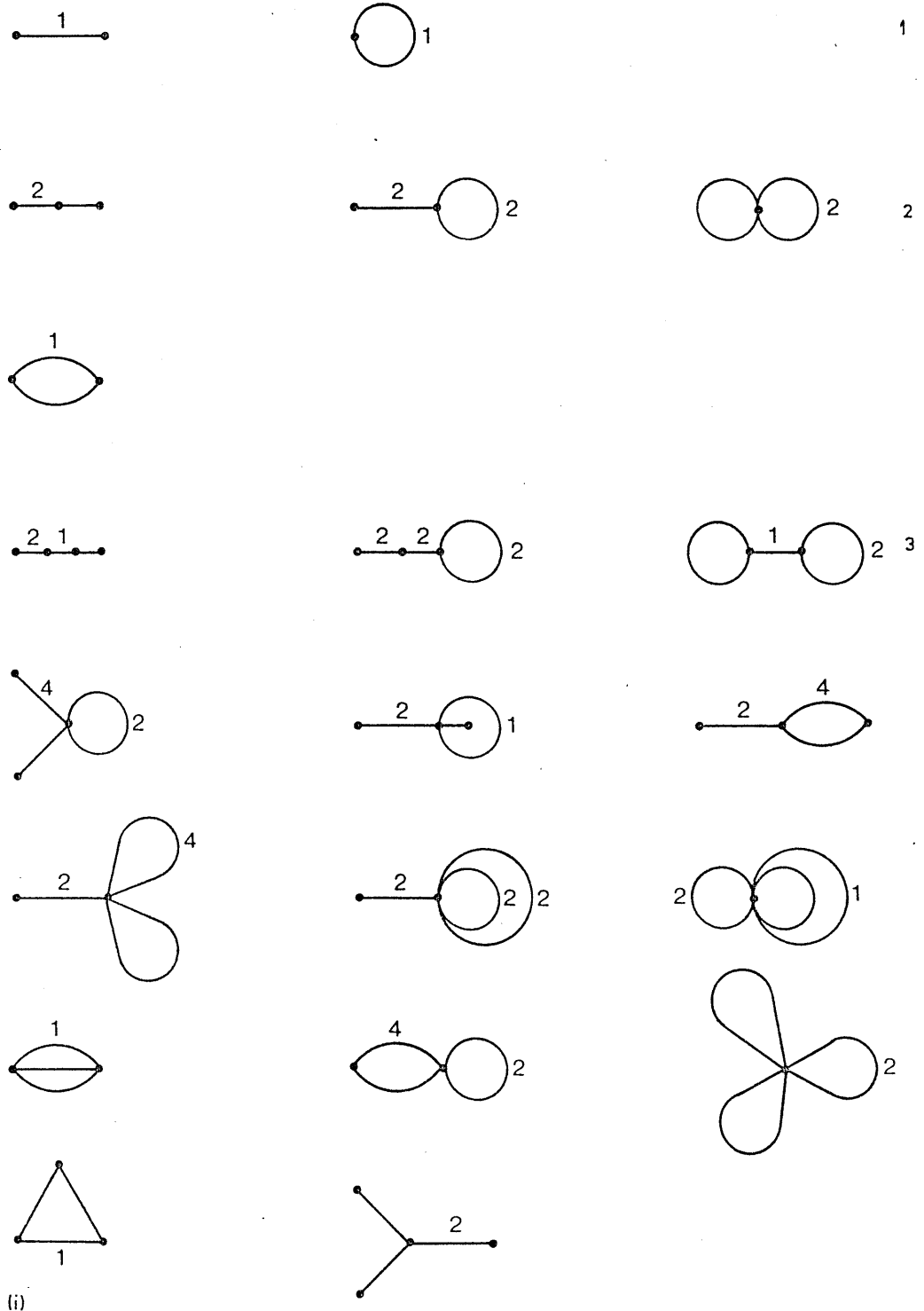
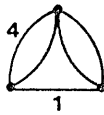
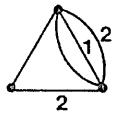
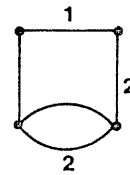
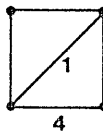
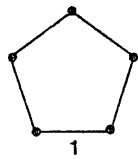
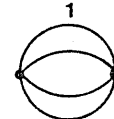
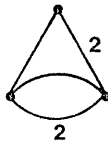
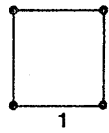
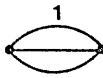
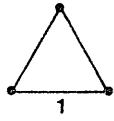
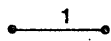
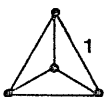
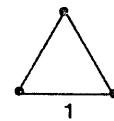
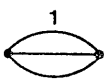
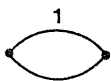


FIGURE 2.5 Rooted plane maps with few edges: (i) $n \leq 3$ edges, (ii) 2-vertex connected, $n \leq 5$, (iii) 3-vertex connected, $n \leq 6$. A number on an edge denotes the number of rooted versions with that edge as root edge.



(ii)



(iii)

FIGURE 2.5

n

1

2

3

4

5

2/3

6

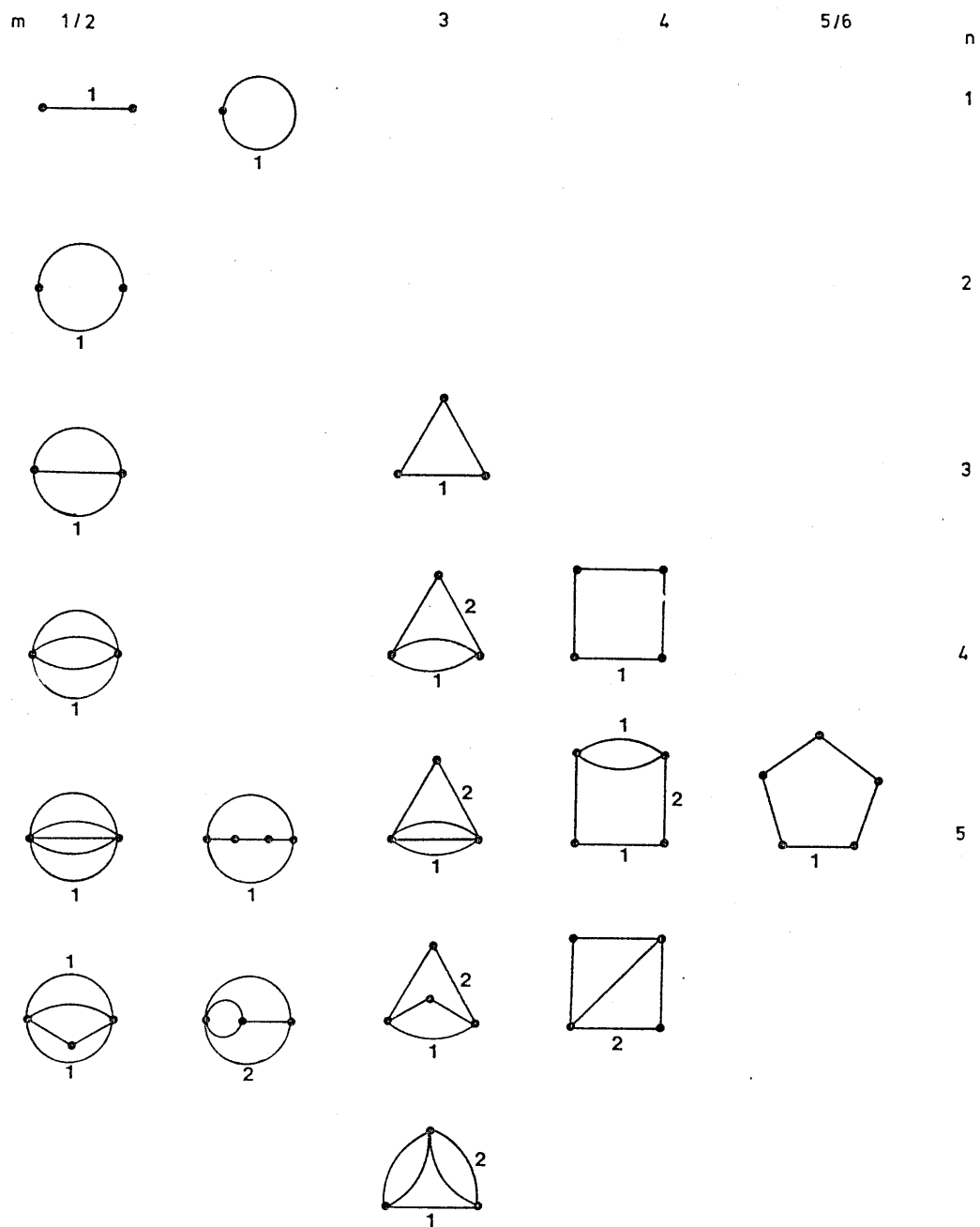


FIGURE 2.6 2-vertex connected rooted plane maps with $n \leq 6$ edges and $m \leq 6$ edges in the boundary of the distinguished face. A number on an edge in the boundary of the distinguished face denotes the number of rooted versions with that edge as root edge.

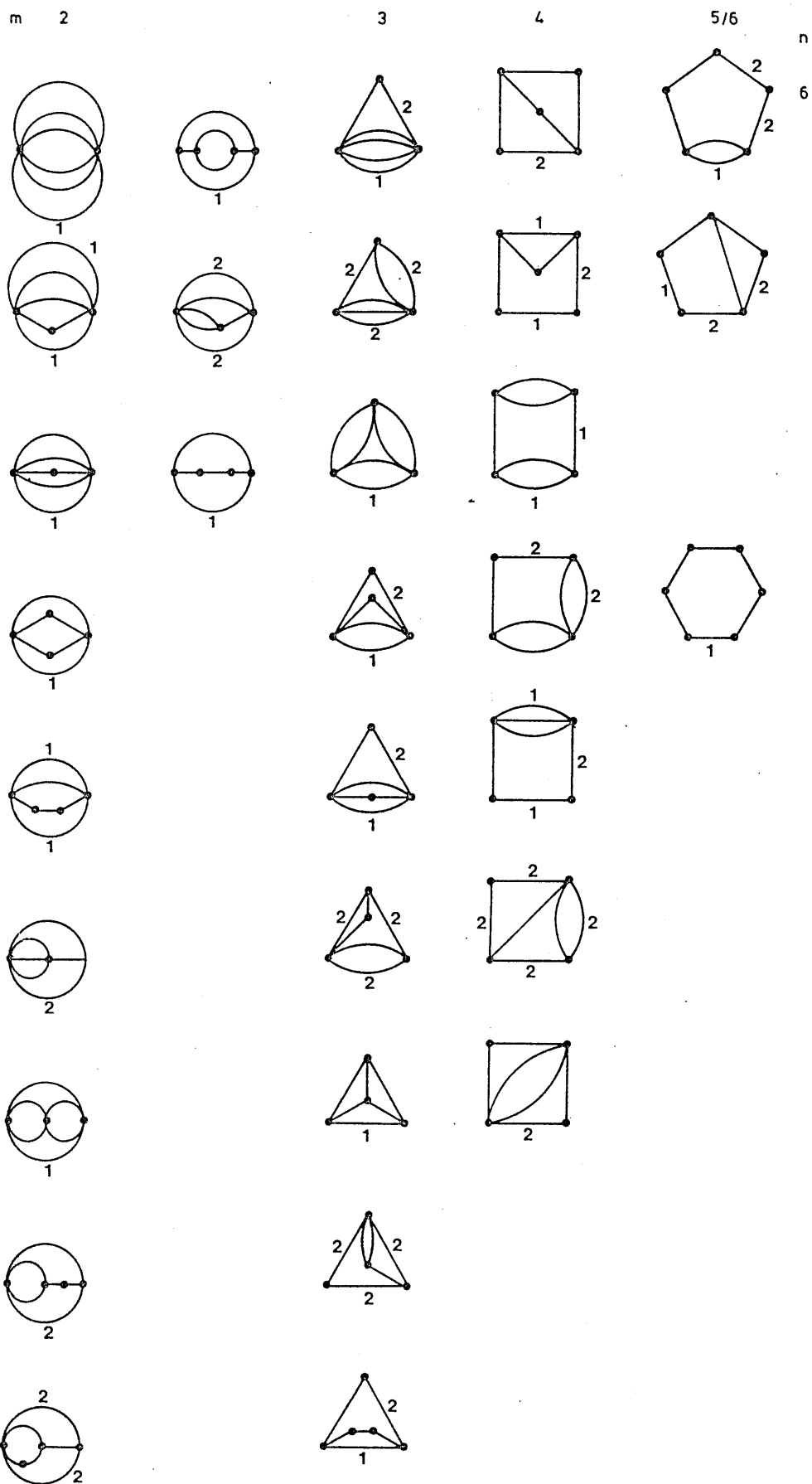


FIGURE 2.6

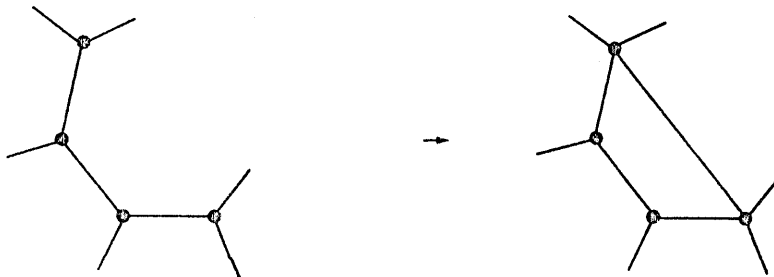


FIGURE 2.7 Operations to construct 3-vertex connected planar maps from wheels.

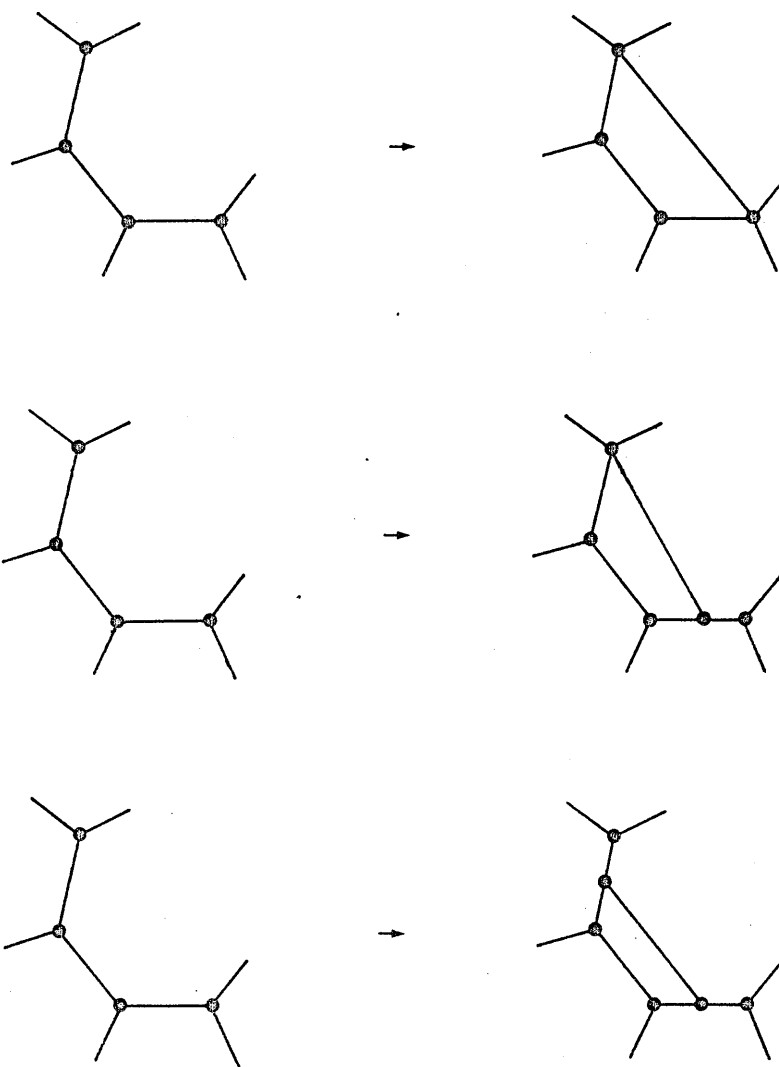


FIGURE 2.8 Operations to construct 3-vertex connected planar maps from the tetrahedron map.

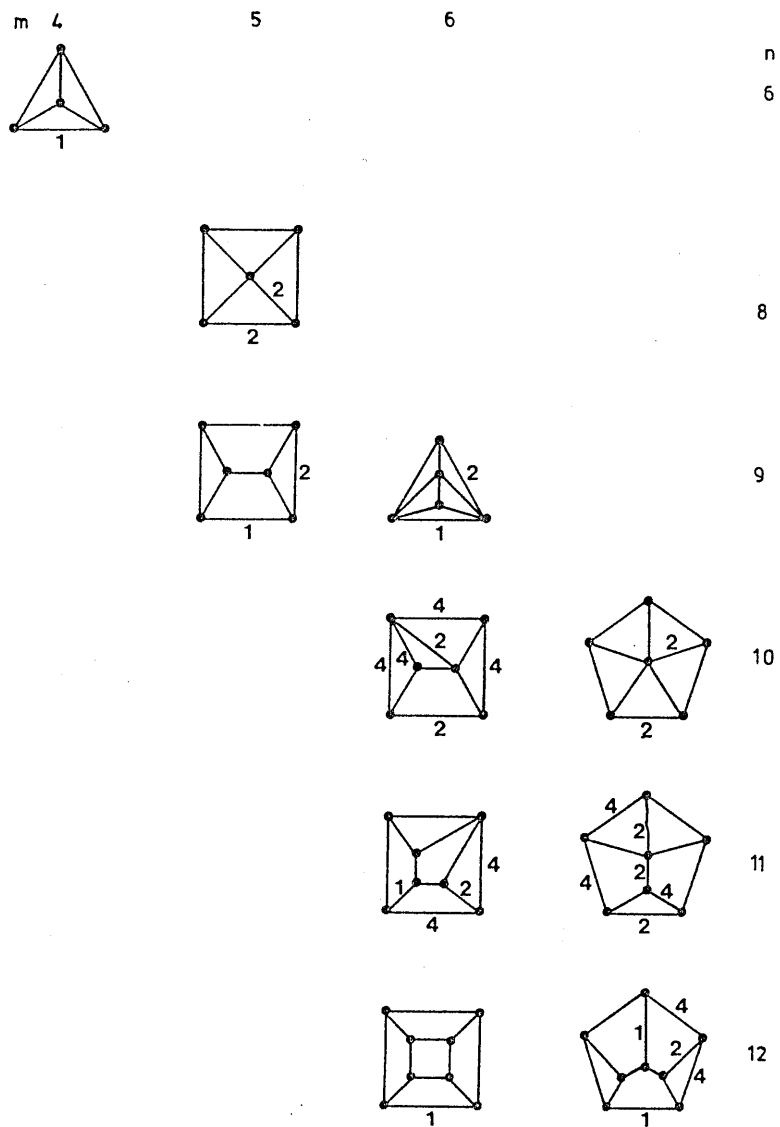


FIGURE 2.9 3-vertex connected planar maps with $m \leq 7$ faces and $n \leq 15$ edges. A number on an edge denotes the number of rooted versions with that edge as root edge.

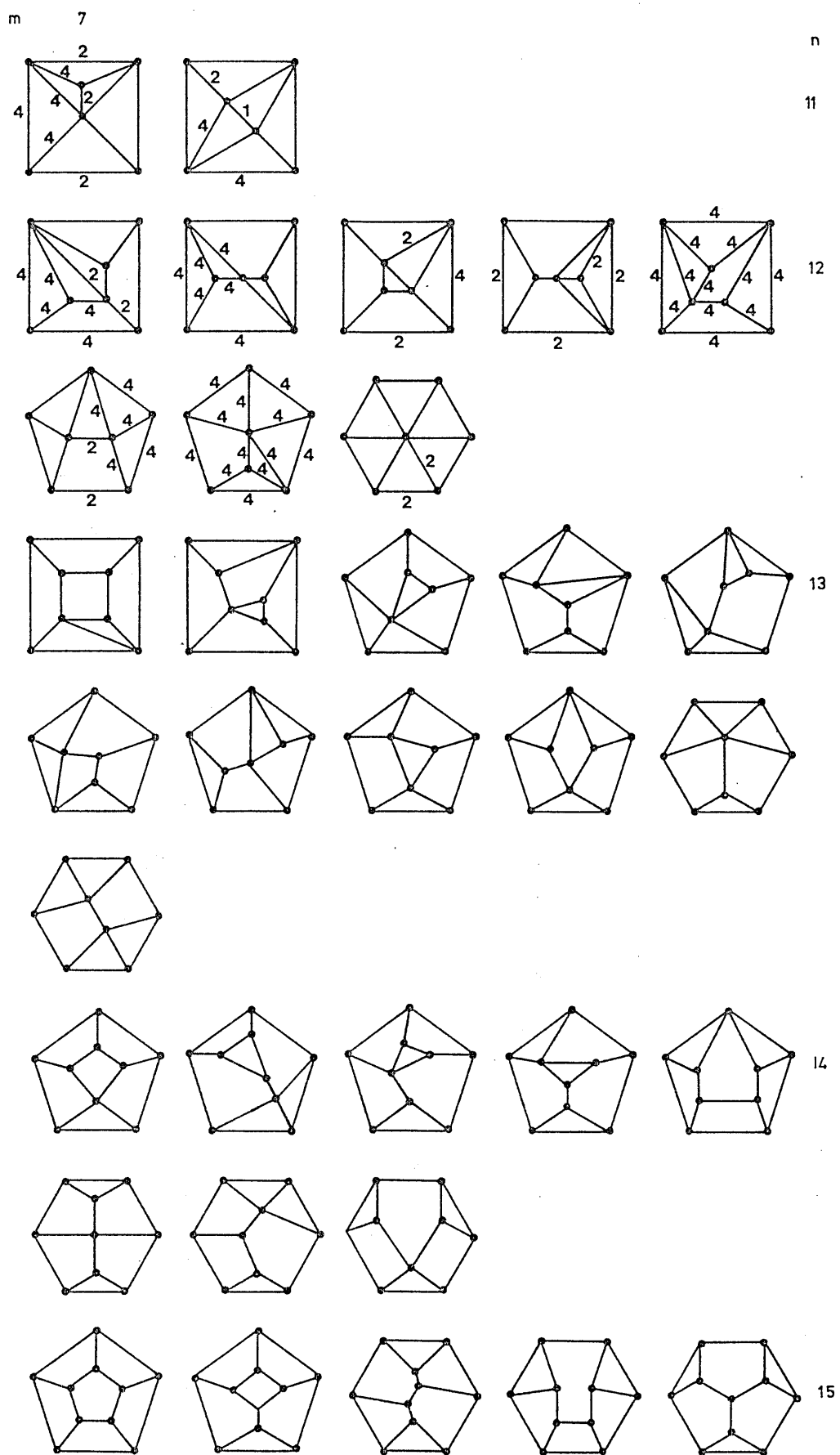


FIGURE 2.9

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n	2	9	54	378	2916	24057								
B_n	2	1	2	6	22	91	408	1938	9614	49335				
C_n	2	1	2	0	0	0	0	4	6	24	66	214	676	2209

TABLE 2.1 1, 2 and 3-vertex connected rooted plane maps with n edges; A_n , B_n and C_n respectively.

n/m	2	3	4	5	6
2	1				
3	1	1			
4	2	3	1		
5	5	9	6	1	
6	22	32	26	10	1

TABLE 2.2 2-vertex connected rooted plane maps with $n \leq 6$ edges and $m \leq 6$ edges in the boundary of the distinguished face.

n/m	2	3	4	5	6
2	1				
3	1	1			
4	2	1	1		
5	4	3	2	1	
6	14	12	8	2	1

TABLE 2.3 2-vertex connected plane maps up to orientation-preserving boundary homeomorphisms with $n \leq 6$ edges and $m \leq 6$ edges in the boundary of the distinguished face.

m/n	6	7	8	9	10	11	12	13	14	15	16
4	1										
5	0	0	4	3							
6	0	0	0	3	24	33	13				
7	0	0	0	0	0	33	188	338	252	68	0
8	0	0	0	0	0	0	13	338	1705	3580	3740
9	0	0	0	0	0	0	0	0	252	3580	16980
10	0	0	0	0	0	0	0	0	0	68	3740

TABLE 2.4 3-vertex connected rooted plane maps with $n \leq 16$ edges and $m \leq 10$ faces.

m/n	6	7	8	9	10	11	12	13	14	15	16	17	18
4	1												
5	0	0	1	1									
6	0	0	0	1	2	2	2						
7	0	0	0	0	0	2	8	11	8	5			
8	0	0	0	0	0	0	2	11	42	74	76	38	14

TABLE 2.5 3-vertex connected planar maps with $n \leq 18$ edges and $m \leq 8$ faces.

3.1 Trivalent maps

In the last chapter the classification of possible architectural plans by the connectedness of their corresponding planar maps was considered. The 3-vertex connected planar maps were a basic set in the sense that other planar maps are constructed from them by adding other maps using 1-vertex or 1-edge extension. This analysis presented the general structure of the space of planar maps, which provides all possibilities for the spatial arrangements in architectural plans. In this chapter a particular class of plans is examined. This class provides both a basic set from which other plans can be constructed by various operations, and a relevant set in that the elements, with perhaps slight modification, correspond to spatial arrangements which commonly occur in architectural plans.

The walls in an architectural plan are most often incident in groups of two, three and four. Suppose that four mutually incident walls are considered as a limiting case of two groups of three incident walls. Consider the corresponding planar maps which have vertices of degree one, two and three. These planar maps are derived from those maps with vertices of degree one and three by edge subdivisions. In turn these are derived from maps with vertices of degree three by 1-augmented 1-vertex extensions by single edges. Planar maps with all vertices of degree three are called trivalent planar maps.

3.1.1 Classification by connectedness

The most general trivalent planar maps have loops and isthmuses.

Consider the rooted plane maps with all vertices degree three except the root vertex which is degree one. Let these maps be called near trivalent rooted plane maps (figure 3.1). There are two trivalent maps with three edges (and none with fewer edges) (figure 3.2) and let the one with loops be called the dumbbell map.

Note that loops in a trivalent map can only occur in conjunction with isthmuses. Also note that $(v;f)$ is a 1-vertex separator in a trivalent planar map if and only if there is a 1-edge separator $(e;f)$ such that v is an endpoint of e . In addition $(v_1, v_2; f_1, f_2)$ is a 2-vertex separator in a trivalent planar map if and only if there is a 2-edge separator $(e_1, e_2; f_1, f_2)$ such that v_1 and v_2 are endpoints of e_1 and e_2 respectively. Thus for trivalent planar maps 2 and 3-vertex connectedness are equivalent to 2 and 3-edge connectedness respectively; vertex and edge connectedness are not distinguished.

Any trivalent rooted plane map can be constructed from a 2-connected trivalent rooted plane map or a rooted dumbbell map by a sequence of 1-augmented 1-vertex extensions by near trivalent rooted plane maps.

On account of the construction of general trivalent planar maps attention is restricted to 2 and 3-connected trivalent planar maps. A 2-connected trivalent rooted plane map can be constructed from a 3-connected trivalent rooted plane map by a unique sequence of 2-augmented 1-edge extensions at distinct non-root edges of the original map. Note that trivalency avoids the difficulties of unique construction encountered for general 2-vertex connected rooted plane maps. This result gives a relation between the generating

functions for the numbers of 2 and 3-connected trivalent rooted plane maps, respectively, with n faces. Consider the general application of 2-augmented 1-edge extension. A 2-connected trivalent rooted plane map can be constructed by a sequence of 2-augmented 1-edge extensions from a 3-connected trivalent rooted plane map using 3-connected trivalent rooted plane maps.

The dual rooted plane maps corresponding to the 3-connected trivalent rooted plane maps have each face bounded by three distinct edges and have no multiple edges. They are rooted triangulations. The number of rooted triangulations with $n \geq 3$ vertices, that is, the number of 3-connected trivalent rooted plane maps with $n \geq 3$ faces is given (Tutte 1962a; Brown 1964) (table 3.1) as

$$\frac{2 (4n - 11)!}{(n - 2)! (3n - 7)!} \quad (1)$$

The 3-connected trivalent planar maps up to homeomorphism have been enumerated up to twelve faces by Grace (1965). Bowen and Fisk (1967) enumerate the corresponding triangulations up to homeomorphism (table 3.2).

The number of 2-connected trivalent rooted plane maps with $n \geq 3$ faces is derived (Tutte 1962b) (table 3.3) as

$$\frac{2^{n-2} 3! (3n - 7)!}{(n - 3)! (2n - 2)!} \quad (2)$$

The duals of the 2-vertex connected trivalent maps are triangular maps in which each face is bounded by three distinct edges, but there may be multiple edges.

3.1.2 Further classification

Consider the general application of 2-augmented 1-edge extension. A 2-connected trivalent rooted plane map can be constructed by a sequence of 2-augmented 1-edge extensions from a 3-connected trivalent rooted plane map using 3-connected trivalent rooted plane maps. Each 2-augmented 1-edge extension creates a corresponding multiple edge in the dual rooted plane map. Consider the two classes of 2-connected trivalent rooted plane maps formed when each 2-augmented 1-edge extension occurs at an edge.

- (1) in the boundary of the distinguished face of the previous map, and
- (2) not in the boundary of the distinguished face of the previous map.

In order to consider these two classes the weak dual rooted plane map of a 2-connected rooted trivalent plane map M is defined. Consider the map $M^* - v^*$, where vertex v^* corresponds to the distinguished face in M , obtained from M^* by deleting v^* and its incident edges and distinguishing the new face formed by the union of v^* and its incident edges and faces. The edge in the distinguished face of M^* , not incident to v^* is the root edge of $M^* - v^*$ with direction given by the orientation of the distinguished face in M^* . This rooted plane map is the weak dual rooted plane map of M (figure 3.3) and it characterises M^* and thus M completely.

The weak dual rooted plane map of a 3-connected trivalent rooted plane map M is a rooted triangulation of a polygon if M has more than three faces, and a single edge if M has three faces. If there are $s \geq 3$ external vertices in the boundary of the distinguished face and r internal vertices it is called a rooted $[r,s]$ triangulation.

The weak dual rooted plane map of a 2-connected trivalent rooted plane map in class (2) is called a rooted $[r,s]$ triangular map if there are $s \geq 2$ external vertices in the boundary of the distinguished face and r internal vertices. The weak dual of a 2-connected trivalent rooted plane map of class (1) can be constructed from a rooted $[r,s]$ triangulation or single edge, by 1-vertex extensions using single edges or rooted $[r,s]$ triangulations which lie in the distinguished face of the previous map.

The weak dual rooted plane map of a general 2-connected trivalent rooted plane map can be constructed from a rooted $[r,s]$ triangular map or single edge by a sequence of 1-vertex extensions by single edges or rooted $[r,s]$ triangular maps which lie in the distinguished face of the previous map.

The $[r,s]$ triangulations represent architectural plans with s regions having an external aspect and r internal regions. There are no "through" rooms or corridors and no two regions which together with their mutually adjacent walls form a multiply connected region, that is, form a "ring".

The $[r,s]$ triangular maps represent architectural plans with s regions having an external aspect and r internal regions. There are no "through" rooms or corridors by virtue of the conditions imposed on the maps of class (2). However, there may be two regions which form a "ring".

The 2-connected trivalent maps of class (1) represent the architectural plans with no two regions which form a "ring". However, there may be "through" rooms or corridors. The 1-vertex

separators in the weak dual map represent the corridors. These may have many adjacencies to the external region.

3.1.3 Enumeration

The number of rooted $[r,s]$ triangular maps, $r \geq 0$, $s \geq 3$, is given by Mullin (1965) as

$$\frac{2^{r+1} (2s-3)! (3r+2s-4)!}{(s-2)!^2 r! (2r+2s-2)!} \quad (3)$$

The number of rooted $[r,2]$, $r \geq 1$, triangular maps is the same as the number of rooted $[r-1, 3]$ triangular maps. The number of rooted $[r,s]$ triangulations is $r \geq 0$, $s \geq 3$, is given by Brown (1964) (table 3.4 and figure 3.4) as

$$\frac{2(2s-3)! (4r+2s-5)!}{(s-1)! (s-3)! r! (3r+2s-3)!} \quad (4)$$

The numbers of $[r,s]$ triangulations up to orientation preserving boundary homeomorphisms are provided by the results in Brown (1964) (table 3.5). The numbers, up to boundary homeomorphisms can also be derived (table 3.6 and figure 3.5) together with the numbers of $[r,s]$ triangulations which possess a reflection symmetry (table 3.7 and figure 3.6). Note that for relatively large numbers of regions only a small proportion of the architectural plans corresponding to 3-connected trivalent plane maps have 'potential' reflection symmetry. This gives some indication of the restriction on possible arrangements imposed if an axis of symmetry is required. For small numbers of regions the restriction is not great.

3.1.4 Simple and strong triangulations

A diagonal in an $[r,s]$ triangulation is an edge not lying in

the boundary of the distinguished face, but whose endpoints both lie in the boundary of the distinguished face. An $[r,s]$ triangulation without diagonals is strong (Tutte 1962a). The corresponding trivalent plane maps have no three mutually adjacent faces (one of which is the distinguished face) whose union together with mutually incident edges and vertices form a multiply connected region. The corresponding architectural plans have no two regions whose union constitutes a "through" space across the plan.

An $[r,s]$ triangulation is simple if every 3-cycle bounds a single face. The trivalent plane maps which correspond to simple $[r,s]$ triangulations, $s \geq 4$, have no three mutually adjacent faces (none of which is the distinguished face) whose union together with mutually incident edges and vertices form a multiply connected region. The simple and strong, simple triangulations are particularly important in the examination of architectural plans in which regions have rectangular boundary (section 3.4). The simple $[r,s]$ triangulations have been investigated by Mullin (1965) who gives an enumeration formula for rooted simple $[r,s]$, $s \geq 4$, triangulations (table 3.8 and figure 3.7) as

$$\frac{(2s - 4)! (3r + s - 4)!}{r! (s - 4)! (s - 1)! (2n + s - 2)!} \quad (5)$$

3.2 The construction of 2 and 3-connected trivalent maps

3.2.1 2-connected trivalent maps

If M is a 2-connected trivalent planar map with $n + 1$ faces then M can be constructed from a 2-connected trivalent planar map with $n \geq 4$ faces by the operation α (figure 3.8) or "face splitting", which adds an edge across the interior of a face. This result

follows immediately from the fact that for each 2-vertex connected graph, not a cycle, there is an edge or suspended chain whose removal yields another 2-vertex connected graph (Whitney 1932).

If M is a 3-connected trivalent planar map with $n + 1$ faces then M can be constructed from a 3-connected trivalent planar map with $n \geq 4$ faces by face splitting.

There are slightly more restricted operations which can be used to generate the trivalent planar maps. A 2-connected trivalent planar map with $f(k)$ faces of degree k has

$$\sum (6 - k) f(k) = 12 \quad . \quad (6)$$

The formula yields the inequality:

$$4f(2) + 3f(3) + 2f(4) + f(5) \geq 12 \quad . \quad (7)$$

Thus a trivalent planar map has a face of degree less than six.

Moreover, a trivalent plane map has an "internal" face; that is a face not the distinguished face, of degree less than six.

Consider the operations shown in figure 3.9. The operations α_i create faces of degree i and are particular cases of the operations α . A 2-connected trivalent planar map with $n + 1$ faces can be constructed from a 2-connected trivalent planar map with $n \geq 4$ faces by $\alpha_2, \alpha_3, \alpha_4$ or α_5 . Further, a 2-connected trivalent plane map with $n + 1$ faces can be constructed from one with n faces by $\alpha_2, \alpha_3, \alpha_4$ or α_5 , such that the new edge does not lie in the boundary of the distinguished face.

3.2.2 3-connected trivalent maps

A 3-connected trivalent planar map with $n + 1$ faces can be

constructed from a 3-connected trivalent planar map with $n \geq 4$ faces by α_3, α_4 , or α_5 . Further, a 3-connected trivalent plane map with $n + 1$ faces can be constructed from one with $n \geq 4$ faces by α_3, α_4 , or α_5 such that the new edge does not lie in the boundary of the distinguished face. The generation of triangular maps and triangulations can be accomplished by using operations β_i dual to $\alpha_i, i = 2, 3, 4, 5$ (figure 3.10).

The weak duals of the 3-connected trivalent plane maps are the $[r, s]$ triangulations. Each triangulation corresponds to a unique 3-connected trivalent plane map. The rules β_3, β_4 and β_5 may be augmented to construct $[r, s]$ triangulations. Consider the operations $\gamma_i, i = 1, \dots, 6$, shown in figure 3.11 where edges marked "p" are in the boundary polygon.

An $[r, s]$ triangulation $T_{r,s}, r \geq 1, s \geq 4$ can be constructed from an $[r - 1, s]$ triangulation by $\beta_3, \beta_4, \beta_5, \gamma_4, \gamma_5$ or an $[r, s - 1]$ triangulation by $\gamma_1, \gamma_2, \gamma_3$. If $T_{r,s}$ has an internal vertex degree three, four or five, then operations $\beta_3, \beta_4, \beta_5$ can be used. Otherwise $T_{r,s}$ has an external vertex v degree two, three or four. If $d(v) = 2$ then use γ_1 . If $d(v) = 3$ (figure 3.12(ii)) and there is no edge (a, c) , then use γ_2 . If there is an edge (a, c) then b is an internal vertex and, use γ_4 . If $d(v) = 4$, then at most one of the pairs of vertices $\{a, c\}$ and $\{b, d\}$ are joined by an edge. If there is no edge (a, d) , then use γ_3 and if there is an edge (a, d) then use γ_5 since b and c are both internal vertices.

An $[r, 3]$ triangulation, $r \geq 1$ can be constructed from an $[r - 1, 3]$ triangulation by $\beta_3, \beta_4, \beta_5$, since it has an internal vertex degree three, four or five.

A strong $[r, s]$ triangulation, $T_{r,s}$, $r \geq 1$, $s \geq 4$ can be constructed from an $[r - 1, s]$ strong triangulation by $\beta_3, \beta_4, \beta_5$ or an $[r, s - 1]$ strong triangulation by γ_2, γ_3 . Suppose first that $T_{r,s}$ has an internal vertex v of degree three, four or five (figure 3.12(i)). If $d(v) = 3$ then β_3 is used. If $d(v) = 4$, then at most one of the pairs of vertices $\{a, c\}$ and $\{b, d\}$ are joined by an edge. If there is an edge (a, c) , then either b or d is an internal vertex and β_4 can be used. If neither is joined by an edge, then β_4 can be used except when a, b, c, d are external vertices. Then $r = 1$, $s = 4$ and $T_{r,s}$ can be constructed using γ_2 . If $d(v) = 5$, then at most two of the pairs of vertices $\{a, c\}$, $\{a, d\}$, $\{b, e\}$, $\{b, d\}$, $\{c, e\}$ are joined by an edge. If there are one or two edges, then at least one of a, b, c, d, e is an internal vertex and β_5 can be used. If there are no such edges, then β_5 can be applied unless a, b, c, d, e are all external vertices, in which case γ_2 is used to construct $T_{r,s}$.

If $T_{r,s}$ has no internal vertex of degree three, four or five, then it has an external vertex v of degree three or four (figure 3.12(ii)). If $d(v) = 3$ then there is no edge (a, c) since $s \geq 4$, thus γ_2 is used. If $d(v) = 4$, then there is no edge (a, d) and at most one of the pairs of vertices $\{a, c\}$ and $\{b, d\}$ are joined by an edge. Thus γ_3 is used to construct $T_{r,s}$.

A strong $[r, 3]$ triangulation, $r \geq 1$ can be constructed from an $[r - 1, 3]$ strong triangulation by $\beta_3, \beta_4, \beta_5$, since it contains an internal vertex of degree three four or five.

A simple $[r, s]$ triangulation, $T_{r,s}$, $r \geq 1$, $s \geq 5$ can be constructed from an $[r - 1, s]$ simple triangulation by $\beta_4, \beta_5, \gamma_4, \gamma_5$

or an $[r, s - 1]$ simple triangulation by $\gamma_1, \gamma_2, \gamma_3$. Suppose first that $T_{r,s}$ has an internal vertex v of degree four or five (figure 3.12(i)). If $d(v) = 4$, then at most one of the pairs of vertices $\{a, c\}$ and $\{b, d\}$ belongs to a non-trivial 4-cycle, that is a four cycle containing internal vertices. Thus β_4 is used. If $d(v) = 5$, then at most three of the pairs of vertices $\{a, c\}, \{a, d\}, \{b, e\}, \{b, d\}, \{c, e\}$ belong to non-trivial 4-cycles. There is at least one vertex a, b, c, d, e which does not belong to any of these 4-cycles. Thus β_5 is used to construct $T_{r,s}$.

If $T_{r,s}$ has no internal vertex of degree four or five, then it has an external vertex v of degree two, three or four (figure 3.12(ii)). If $d(v) = 2$, then γ_1 is used. If $d(v) = 3$, and the pair of vertices $\{a, c\}$ does not belong to a non-trivial 4-cycle, then γ_2 is used. If $\{a, c\}$ belongs to a non-trivial 4-cycle, then b is an internal vertex and γ_4 is used. If $d(v) = 4$ and the pairs of vertices $\{a, c\}$ and $\{b, d\}$ belong to non-trivial 4-cycles, then b is an internal vertex with $d(b) = 4$, a contradiction. If $\{a, d\}$ belongs to a non-trivial 4-cycle, then γ_5 is used and if not then γ_3 is used to construct $T_{r,s}$.

A simple $[r, s]$ triangulation $r \geq 1, s = 3, 4$ can be constructed from a simple $[r - 1, s]$ triangulation by β_4 or β_5 since it contains an internal vertex degree four or five.

A strong, simple $[r, s]$ triangulation, $T_{r,s}$, $r \geq 1, s \geq 5$ can be constructed from a strong, simple $[r - 1, s]$ triangulation by $\beta_4, \beta_5, \gamma_4, \gamma_5$ or a strong, simple $[r, s - 1]$ triangulation by γ_2, γ_3 . It should be noted that a simple $[r, s]$ triangulation $r \geq 1, s = 3, 4$, is also strong.

3.3 Transformations of trivalent maps

An edge contraction in a planar map consists in removing an edge with disjoint endpoints and then identifying the endpoints. The inverse operation is vertex expansion. Any 2-vertex connected planar map may be constructed from a 2-connected trivalent planar map by a sequence of edge contractions and edge subdivisions. Similarly any 3-vertex connected planar map may be constructed by a sequence of edge contractions from a 3-connected trivalent planar map. The 3-connected trivalent planar maps are considered as a basic or fundamental set of spatial arrangements in architectural plans, (March and Earl 1977, Earl and March 1979.)

An exchange operation which allows 2-connected trivalent planar maps to be constructed from one another, thus allowing local changes to be made whilst preserving trivalency, consists of an edge contraction followed by a vertex expansion (figure 3.13(i)). In the dual planar map exchange corresponds to a diagonal transformation (Ore 1967), (figure 3.13(ii)). Any 2-connected trivalent planar map can be transformed into any other by a sequence of exchange operations.

Consider applications of edge contraction to 3-connected trivalent plane maps such that no edge in the boundary of the distinguished face is contracted and each resulting vertex is degree three or four. Thus the faces adjacent to the distinguished face, which represent regions with external aspect in the corresponding architectural plan, remain adjacent after the edge contractions. These edge contractions on 3-connected trivalent plane maps can be represented by marking the edges in the weak dual $[r,s]$ triangulations. Let an edge be marked "c" if the corresponding edge is contracted.

The boundary of each triangular face has at most one edge marked "c". Suppose that the edge contractions are further restricted so that no edge incident to a vertex in the boundary of the distinguished face is contracted. The corresponding marked $[r,s]$ triangulations have no edge in the boundary of the distinguished face marked "c" and at most one edge in each triangular face marked "c". If the marked edges are removed from the triangulations, the resulting plane maps have internal faces bounded by three or four distinct edges. If all internal faces are degree four quadrangulations (Brown 1965) are obtained.

3.4 Rectangular Maps

The transformations in section 3.3 preserve the types of elements and relations, namely wall and region adjacency, but transform those adjacencies. Other transformations can occur which augment the adjacency structure, perhaps by considering other types of relations between the elements or other properties of the elements.

An example would be the addition of an access or permeability structure to a given spatial arrangement. This requires considering an access relation between regions, and by implication requires discriminating between walls which allow access and which do not allow, or prevent, access. Thus a new relation is considered between elements, and new properties of some elements are considered.

In this section transformations are considered on certain classes of trivalent maps which give them a realization with all faces having a rectangular boundary.

Consider a 3-connected trivalent plane map $M(f)$, whose weak dual is an $[r,s]$ triangulation. Suppose there is a realization of $M(f)$ in the plane with each face, not adjacent to f , bounded by a rectangle. Let such a rectangular realization of $M(f)$ be called an $[r,s]$ rectangular map. The edges of $M(f)$ in such a realization, which are not incident to vertices in the boundary of f are straight line segments parallel to x and y coordinate axes (figure 3.14).

An $[r,s]$ rectangular map represents an architectural plan with r internal rooms which are all rectangles and with the external region divided into s parts. The division of the external region becomes relevant when the plan is given geometrical properties. There are various facades and courtyards around the plan and their relation to the internal spaces are often as important as the relations between the internal spaces themselves.

An $[r,s]$ rectangular map is represented by the weak dual $[r,s]$ triangulation with internal edges coloured x and y corresponding to the two directions of the line segments. Let the boundary edges be coloured z . Call this the dual of the rectangular map (figure 3.15). Two rectangular maps are considered equivalent if there is a boundary homeomorphism of the corresponding duals which preserves or reverses the colours on all internal edges.

The aim of this section is to examine the coloured $[r,s]$ triangulations which correspond to $[r,s]$ rectangular maps. The colourings are then considered as ornamentations of 3-connected trivalent plane maps which represent realizations as rectangular maps.

Consider a coloured $[r,s]$ triangulation which is the dual of

an $[r,s]$ rectangular map. The colouring induces an associated labelling according to the scheme in figure 3.16. The associated labelling satisfies the following conditions.

- 1) Each internal face has labels of sum two.
- 2) Each internal vertex has labels of sum four.

A colouring of an $[r,s]$ triangulation whose associated labelling satisfies (1) and (2) is called a subvalid colouring. The main result (proposition 3.2) of this section demonstrates that the subvalid colourings of $[r,s]$ triangulations are precisely the duals of $[r,s]$ rectangulation maps. However, first note that not every $[r,s]$ triangulation has a subvalid colouring. Proposition 3.1 exhibits those which have subvalid colourings. As a preliminary some properties of subvalid colourings are given.

Consider a subvalidly coloured $[r,s]$ triangulation T . Suppose C is a k -cycle in T which forms a simple closed curve. Consider the associated labelling of T . Let the labels at the vertices of C , which belong to faces inside C , have sum p . If there are n vertices inside C then by counting labels there are $2n + \frac{1}{2}p$ faces inside C . Applying Euler's formula gives $p = 2k - 4$. As corollaries each 3-cycle bounds an interior face, and a non-trivial 4-cycle, that is a 4-cycle with internal vertices, has one of the forms shown in figure 3.17. The internal edges at opposite boundary vertices on a 4-cycle all have the same colour and neighbouring vertices have internal edges of opposite colour. Note that a colouring of an $[r,s]$ triangulation is subvalid if and only if $p = 2k - 4$ for all k -cycles.

Proposition 3.1. An $[r,s]$ triangulation T , has a subvalid colouring if and only if $s \geq 4$ and T is simple.

Proof. First note that each simple $[r,4]$ triangulation has a subvalid colouring since it is the weak dual of a 3-connected trivalent plane map which has a realization as an $[r,4]$ rectangular map (see Earl and March 1979, Theorem 5.3, for $r \geq 1$).

The proof of the proposition is by induction on s . Suppose that the proposition is true for all simple $[r,s]$ triangulations for a given r and $4 < s \leq n$. Consider a simple $[r, n+1]$ triangulation $T(n+1)$. Let (u,v) be an edge in the boundary of $T(n+1)$. If (u,v) does not belong to a non-trivial 4-cycle, then contract (u,v) to obtain a simple $[r,n]$ triangulation which has a subvalid colouring. Reverse the contraction and assign colours as shown in figure 3.18. The result is a subvalid colouring of $T(n+1)$. Now suppose that (u,v) belongs to a non-trivial 4-cycle C . It may be assumed that (u,v) does not belong to a non-trivial 4-cycle, C' , where C consists of vertices and edges on or in the interior of C' . Remove the vertices from the interior of C and contract (u,v) . The result is a simple $[r',n]$, $r' < r$, triangulation which has a subvalid colouring. Reverse the contraction. The 4-cycle C and its interior edges form a simple $[r-r', 4]$ triangulation, which has a subvalid colouring with internal edges at opposite boundary vertices all the same colour. Inserting this subvalidly coloured triangulation, a subvalid colouring of $T(n+1)$ is obtained. The induction is completed by using the fact that each simple $[r,4]$ triangulation has a valid colouring.

If T has a subvalid colouring then the condition on k -cycles above, implies that $k = 3$ if and only if the cycle bounds an internal face. Thus T is simple and $s \geq 4$.

Proposition 3.2. Each subvalid colouring of an $[r,s]$ triangulation

is the dual of an $[r,s]$ rectangular map.

Proof. The proof is by induction on $r + s$. Suppose the proposition holds for all $[r,s]$ triangulations with $r + s \leq n$. Let $T(p,q)$ be a subvalidly coloured $[p,q]$ triangulation with $p + q = n + 1$. Let the vertices in the boundary of $T(p,q)$ be N_1, N_2, \dots, N_q in cyclic order.

Suppose first that there is a diagonal (N_i, N_j) , $|i-j| > 2$, in $T(p,q)$. Decompose $T(p,q)$ into two parts as shown in figure 3.19. The parts are subvalidly coloured $[r_1, s_1]$ and $[r_2, s_2]$ triangulations with $r_i + s_i \leq n$, $i = 1, 2$, and are thus duals of $[r_i, s_i]$ rectangular maps $R(r_i, s_i)$, $i = 1, 2$. The subvalid colourings may be chosen such that $R(r_1, s_1)$ and $R(r_2, s_2)$ can be assembled to form a $[p,q]$ rectangular map $R(p,q)$, whose dual is $T(p,q)$ (figure 3.20). If $|i-j| = 2$ then one of the parts is the $[0,3]$ triangulation, the other is a subvalidly coloured $[p,q-1]$ triangulation which is the dual of a $[p,q-1]$ rectangular map $R(p,q-1)$. A rectangular map $R(p,q)$ whose dual is $T(p,q)$ can be constructed from $R(p,q-1)$ (figure 3.21).

Suppose next that there is a face in $T(p,q)$ with one z and two x or two y edges. The z edge, (N_i, N_{i+1}) , can be contracted to form a subvalidly coloured $[p,q-1]$ triangulation which is the dual of a $[p,q-1]$ rectangular map $R(p,q-1)$. A rectangular map $R(p,q)$ whose dual is $T(p,q)$ can be constructed from $R(p,q-1)$ (figure 3.22).

Now suppose that $T(p,q)$ has no diagonal. There is at least one boundary vertex N_k , say, with incident internal edges all coloured x or y . This follows from the condition on cycles in a subvalidly coloured triangulation. Consider the configurations in figure 3.23 (up to reversal of x and y colours). It is straight-

forward to show that at least one occurs in $T(p,q)$. If (a) occurs then (N_{k-1}, N_k) may be contracted to give a subvalidly coloured $[p,q-1]$ triangulation with edge $(v, N_{k-1} = N_k)$ coloured x . This is the dual of a $[p,q-1]$ rectangular map $R(p,q-1)$. A rectangular map $R(p,q)$ whose dual is $T(p,q)$ can be constructed from $R(p,q-1)$, (figure 3.24).

Suppose (b) but not (a) occurs in $T(p,q)$. First let $u = N_{k-1}$. If v is adjacent to $\{N_{k-1}, N_k\}$ but to no other vertices in the boundary of $T(p,q)$, then (u,v) can be contracted to give a subvalidly coloured $[p-1,q]$ triangulation which is the dual of a $[p-1,q]$ rectangular map $R(p-1,q)$. An operation on $R(p-1,q)$, of the type shown in figure 3.25, yields a $[p,q]$ rectangular map $R(p,q)$ whose dual is $T(p,q)$. If v is adjacent to $\{N_{k-1}, N_k, N_{k+1}\}$ but to no other vertices in the boundary of $T(p,q)$, then one of the configurations in figure 3.26 occurs. In (i) contract edges $(N_{k-1}, v), (v, N_{k+1})$ and remove vertex N_k . In (ii), contract (N_{k-1}, v) and remove N_k . In each case recolour new boundary edges. Subvalidly coloured $[p-1, q-2]$ and $[p-1, q-1]$ triangulations, respectively, are obtained. These are the duals of $[p-1, q-2]$ and $[p-1, q-1]$ rectangular maps $R(p-1, q-2)$ and $R(p-1, q-1)$. The operations in figure 3.27, on $R(p-1, q-2)$ and $R(p-1, q-1)$, yield $[p,q]$ rectangular maps $R(p,q)$ whose duals are $T(p,q)$. Finally, if v is adjacent to some N_i , $i \neq k-1, k, k+1$ and (v, N_i) is coloured x then decompose $T(p,q)$ as shown in figure 3.28 and add new vertices n_1 and n_2 . The parts are subvalidly coloured $[r_i, s_i]$ triangulations with $r_i + s_i \leq n$, $i = 1, 2$, and are thus duals of $[r_i, s_i]$ rectangular maps $R(r_i, s_i)$, $i=1, 2$. These can be assembled to form a $[p,q]$ rectangular map $R(p,q)$ whose dual is $T(p,q)$ (figure 3.29). If (v, N_i) is coloured y then the argument is similar.

Second, let u and v be internal vertices of $T(p,q)$. The edge (u,v) can be contracted to give a subvalidly coloured $[p-1,q]$ triangulation, which is the dual of a $[p-1,q]$ rectangular map $R(p-1,q)$. A $[p,q]$ rectangular map $R(p,q)$ whose dual is $T(p,q)$ can be constructed from $R(p-1,q)$ by an operation of the type shown in figure 3.30.

Finally suppose (c) occurs in $T(p,q)$. If v is not adjacent to N_i , $i \neq k-1, k, k+1$, then (N_k, v) can be contracted to give a subvalidly coloured $[p-1,q]$ triangulation which is the dual of a $[p-1,q]$ rectangular map $R(p-1,q)$. A (p,q) rectangular map $R(p,q)$ whose dual is $T(p,q)$ can be constructed from $R(p-1,q)$ by an operation of the type shown in figure 3.31. If v is adjacent to some N_i , $i \neq k-1, k, k+1$, then (N_i, v) is coloured x and decompose $T(p,q)$ as shown in figure 3.32, where n_1 and n_2 are new vertices. The components are subvalidly coloured $[r_i, s_i]$ triangulations $r_i + s_i \leq n$, $i = 1, 2$ and are thus duals of $[r_i, s_i]$ rectangular maps $R(r_i, s_i)$, $i = 1, 2$. These can be assembled to form a $[p,q]$ rectangular map $R(p,q)$ whose dual is $T(p,q)$ (figure 3.33).

To complete the induction, the subvalidly coloured $[0,4]$ triangulation is the dual of the rectangular map in figure 3.34.

The proof of proposition 3.2 might be facilitated by the use of the methods of construction for $[r,s]$ triangulations given in section 3.2. The coloured versions of the operations given there may be related to operations on the rectangular maps. This idea can certainly be used to show that each subvalid colouring of an $[r,4]$ triangulation $r \geq 1$ is the dual of an $[r,4]$ rectangular map. However, it seems that in general the number of cases to be considered present major complications.

Some classes of rectangular maps are now examined. A subvalidly coloured $[r,s]$ triangulation has four types of triangular faces with (a) no boundary edges, (b) one boundary edge, one x edge and one y edge, (c) one boundary edge and two x edges or two y edges, or (d) two boundary edges.

The faces of type (b), (c) and (d) represent "corners" in the boundary of the corresponding plan. The respective corners have external angles of $3\pi/2$, π , and 2π (figure 3.35). If there are only faces of type (a) and (b) then the colouring is called valid. The internal edges of the corresponding $[r,s]$ rectangular map form an $[r,s]$ rectangulation. The validly coloured $[r,s]$ triangulation is called the augmented dual of the rectangulation. An $[r,s]$ rectangulation represents an architectural plan with r internal rooms which are all rectangles and with the external region divided into s parts at the $3\pi/2$ corners in the boundary of the plan (figure 3.36(i)).

A valid colouring of an $[r,s]$ strong triangulation corresponds to a rectangulation in which each line segment belongs to the boundary of an internal rectangle. Let these be called strong $[r,s]$ rectangulations (figure 3.36(ii)). They have a boundary which is a right angled polygon.

Finally the $[r,4]$ rectangular maps are considered. Each subvalid colouring of an $[r,4]$ triangulation is valid and the triangulation is strong. It is thus the augmented dual of an $[r,4]$ strong rectangulation. These are the rectangulations in which the boundary is a rectangle. They are referred to simply as rectangulations.

The transformation which gives a valid colouring to a simple $[r,4]$ triangulation, $r \geq 1$, specifies a rectangulation. In order to examine the effect of this ornamentation the numbers of rooted simple $[r,4]$ triangulations and validly coloured rooted simple $[r,4]$ triangulations are compared. The number of validly coloured rooted simple $[r,4]$ triangulations has been given by Flemming (1977) for $r \leq 10$, (table 3.9), by counting the corresponding "fixed" rectangulations using a recurrence based on the fact that each rectangulation has a "corner" rectangle in one of the configurations in figure 3.37. The comparison with the number of rooted simple $[r,4]$ triangulations (Mullin 1965), (table 3.10), suggests that for small numbers of rectangles, namely $r \leq 6$, the colouring transformation cannot be applied in many different ways for a given triangulation. However, as r increases, the choices presented by the colouring transformation increase substantially.

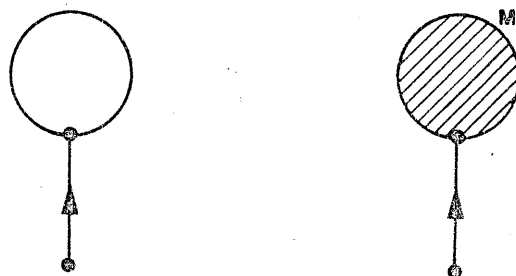


FIGURE 3.1 Near trivalent rooted plane maps. M denotes a 2-vertex connected trivalent plane map.

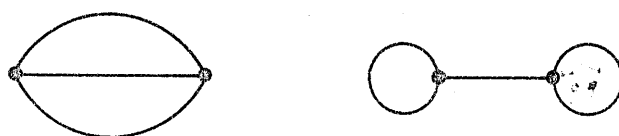


FIGURE 3.2 Trivalent planar maps with three edges.

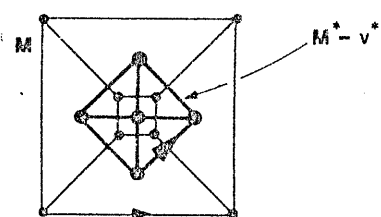
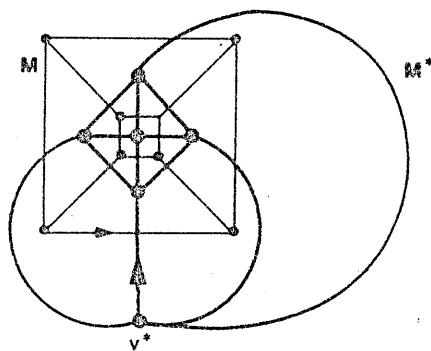


FIGURE 3.3 Weak dual rooted plane map

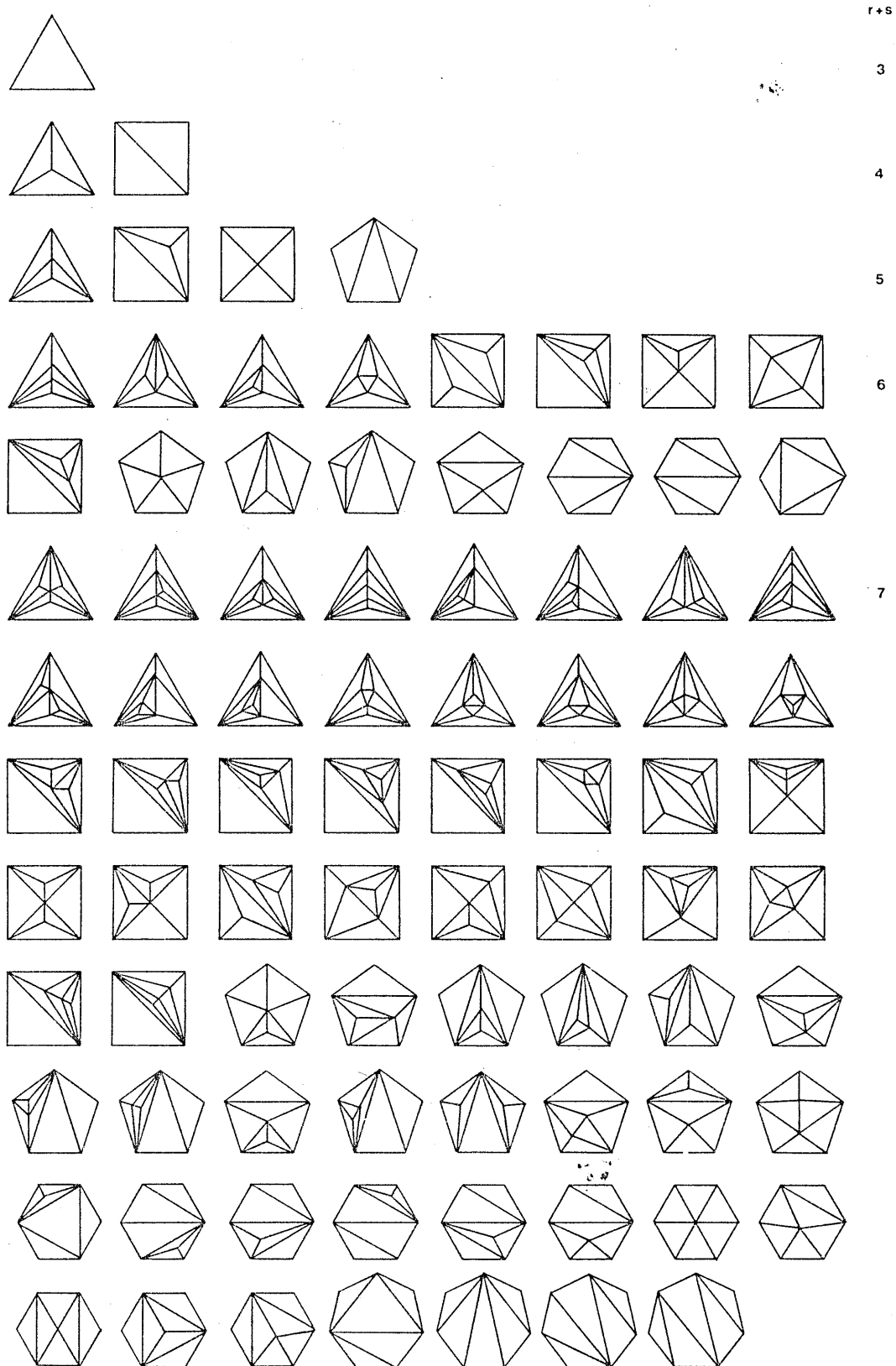


FIGURE 3.5 $[r,s]$ triangulations, $r + s \leq 7$, up to boundary homeomorphisms.

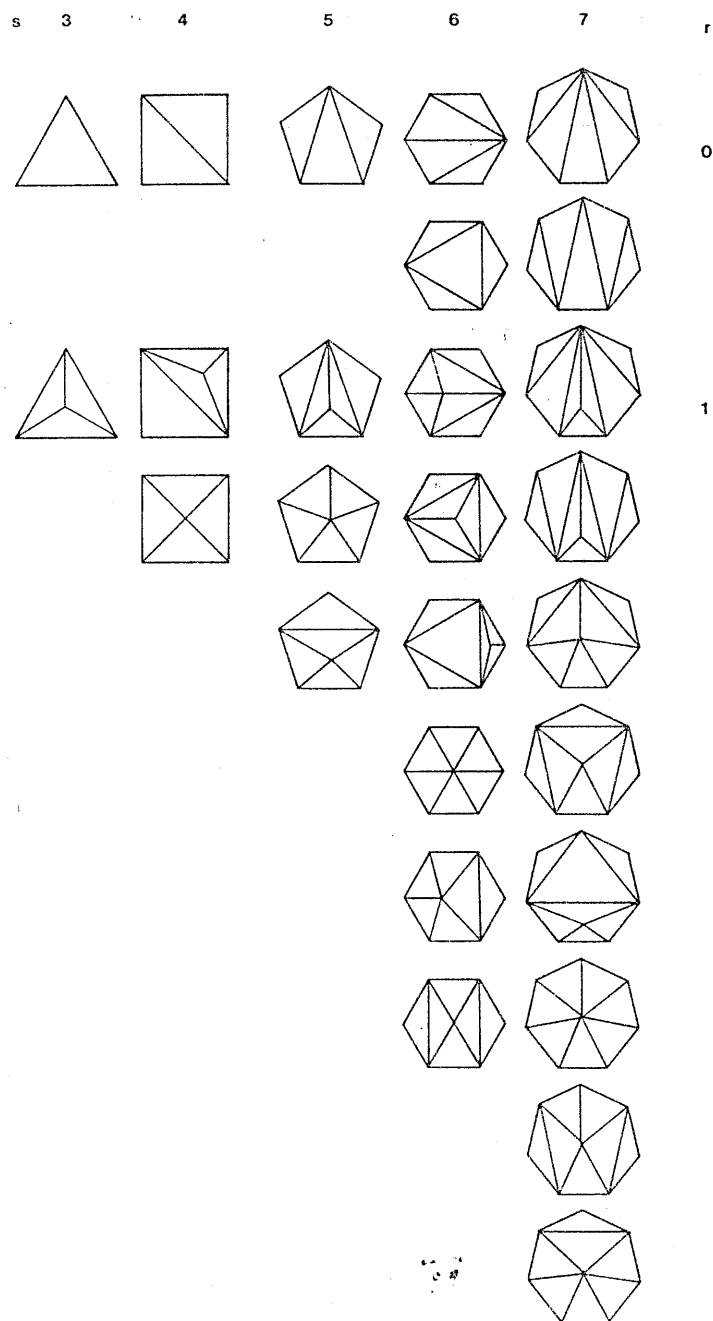


FIGURE 3.6 $[r,s]$ triangulations, $r \leq 1$, $s \leq 7$, which have reflection symmetry.

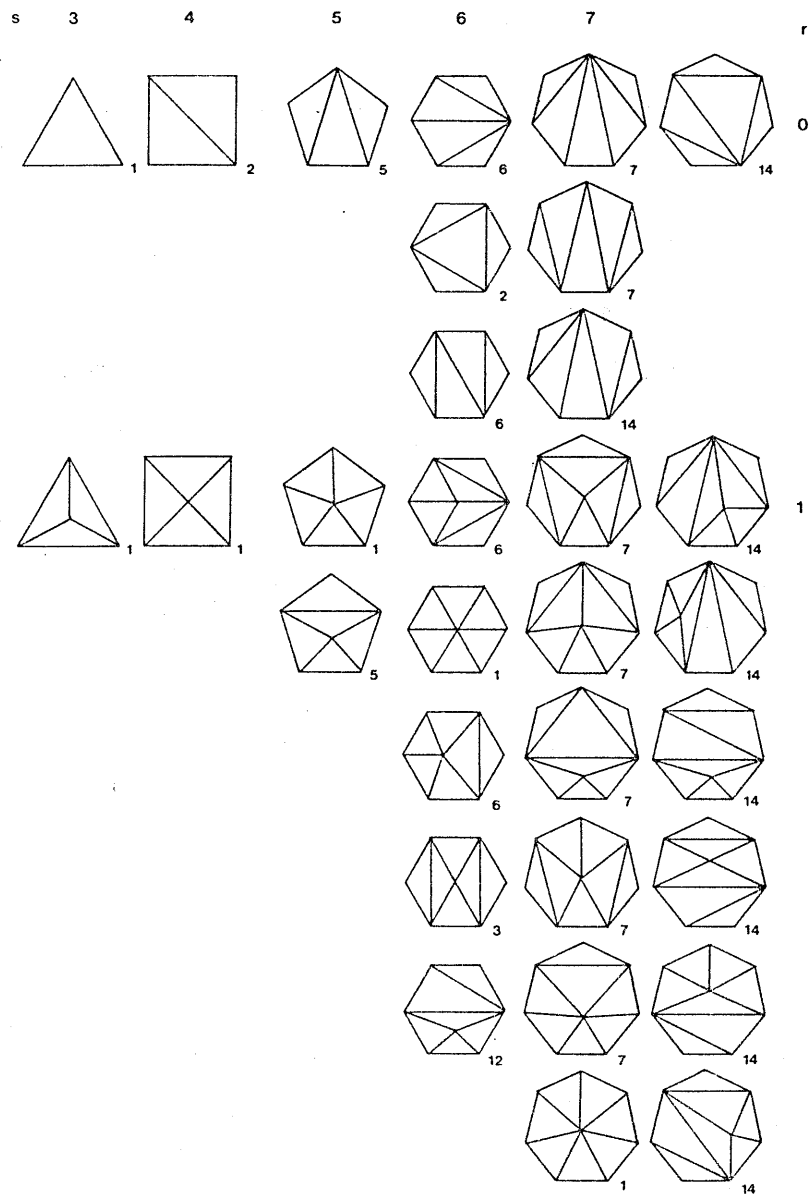


FIGURE 3.7 Simple $[r,s]$ triangulations, $r \leq 1$, $s \leq 7$. A number on a triangulation denotes the number of rooted versions.

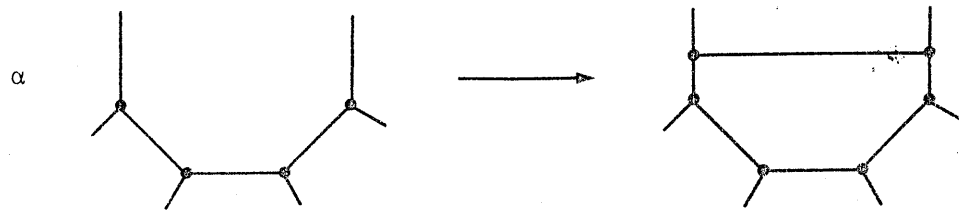


FIGURE 3.8 Face splitting.

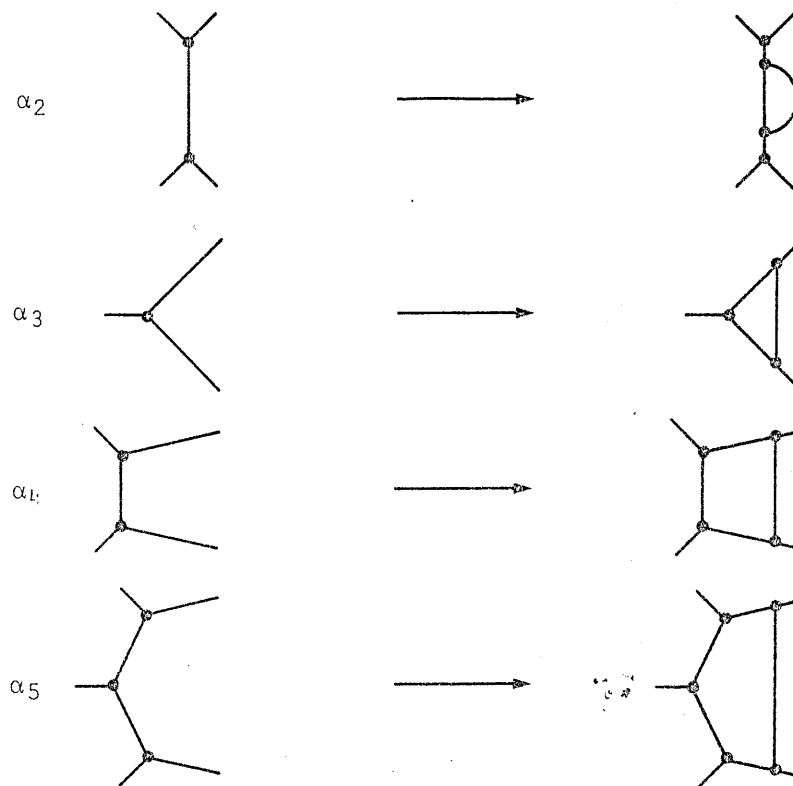


FIGURE 3.9 Particular cases of face splitting, the operations $\alpha_2, \alpha_3, \alpha_4$ and α_5 .

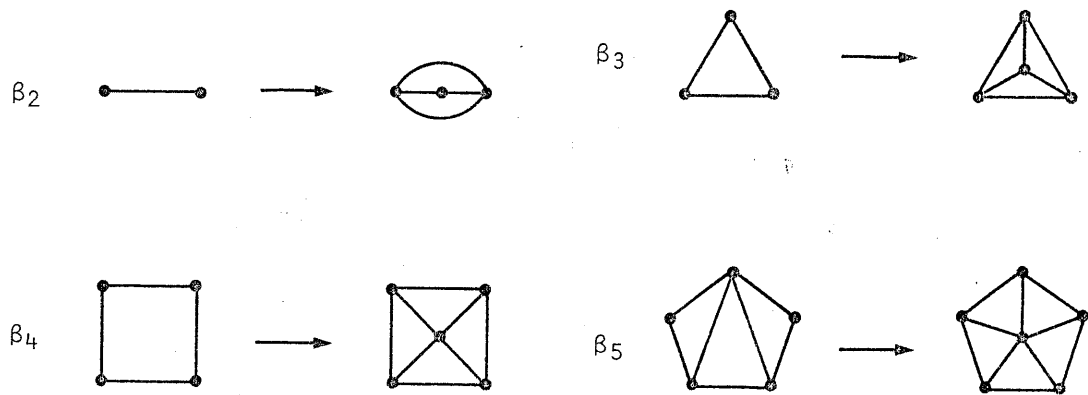


FIGURE 3.10 The operations $\beta_2, \beta_3, \beta_4$ and β_5 dual to $\alpha_2, \alpha_3, \alpha_4$ and α_5 .

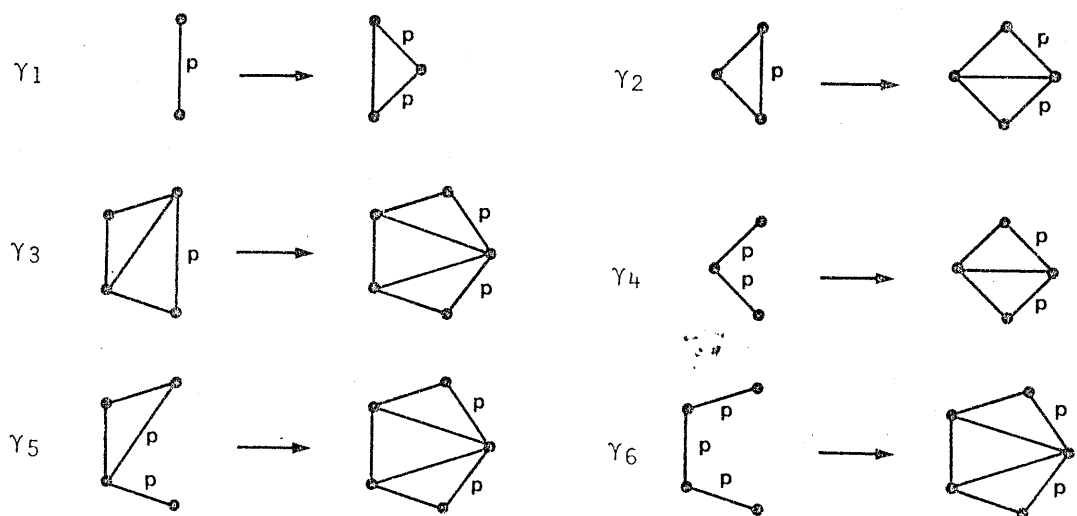
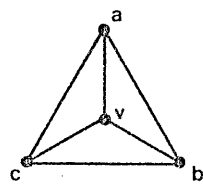
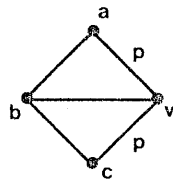
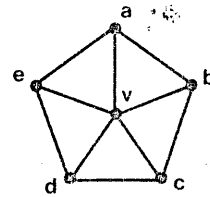
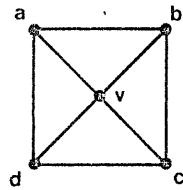


FIGURE 3.11 Operations $\gamma_i, 1 \leq i \leq 6$.



(i)



(ii)

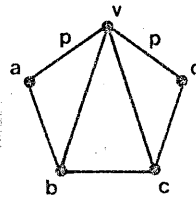
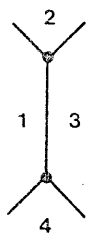
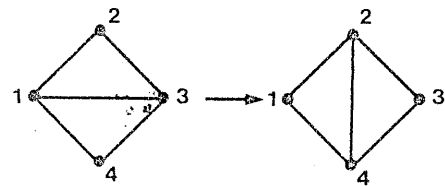
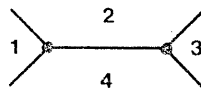


FIGURE 3.12 Vertices in $T_{r,s}$.



(i)



(ii)

FIGURE 3.13 (i) Exchange operation and (ii) diagonal transformation.

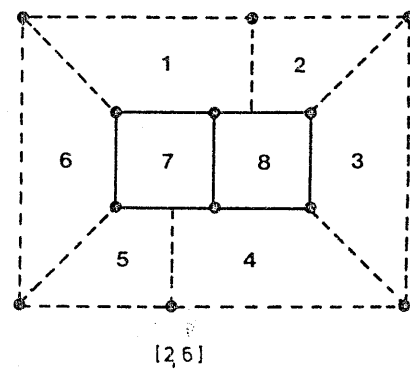
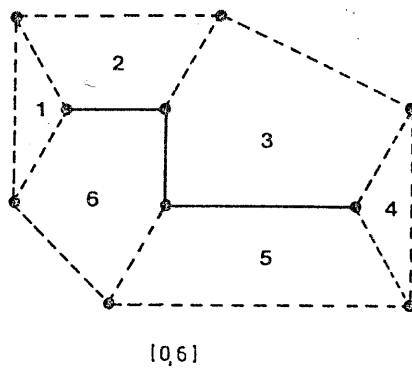


FIGURE 3.14 Rectangular maps.

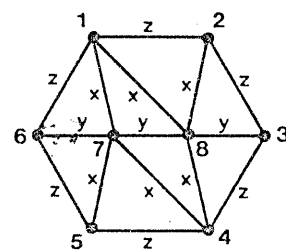
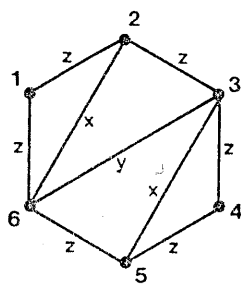


FIGURE 3.15 Duals of rectangular maps in figure 3.14.

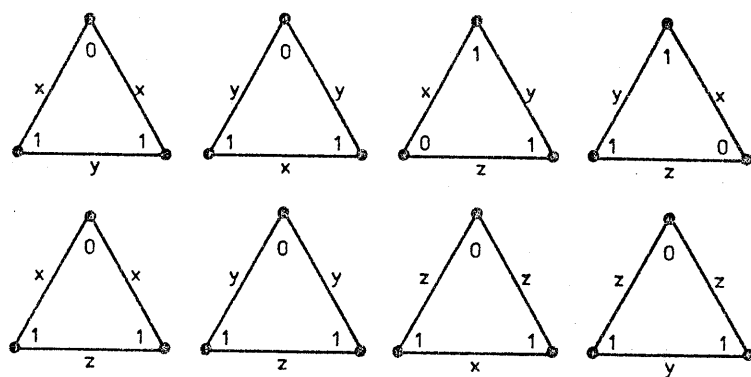


FIGURE 3.16 Face labels associated with subvalid colouring.

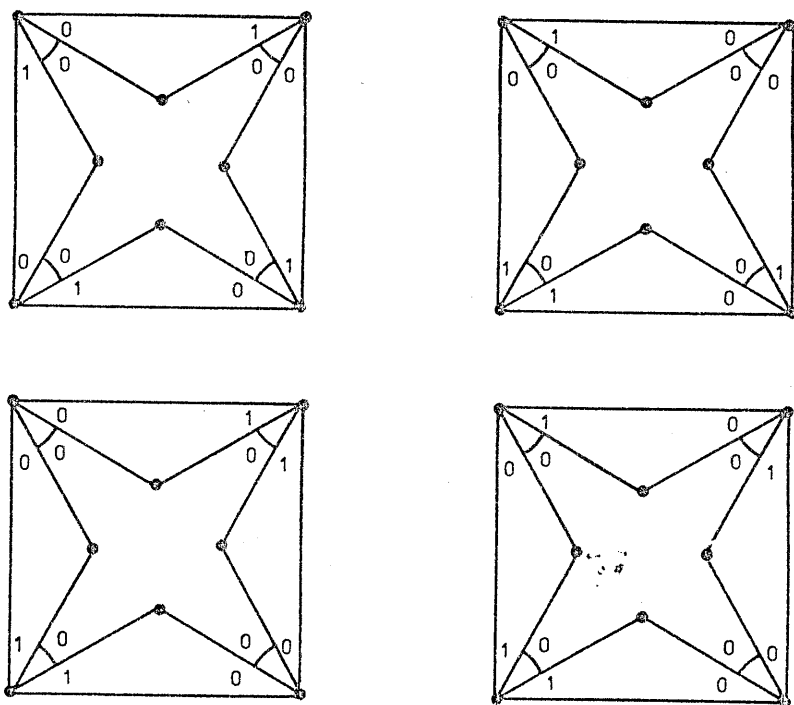


FIGURE 3.17 Non trivial 4-cycles.

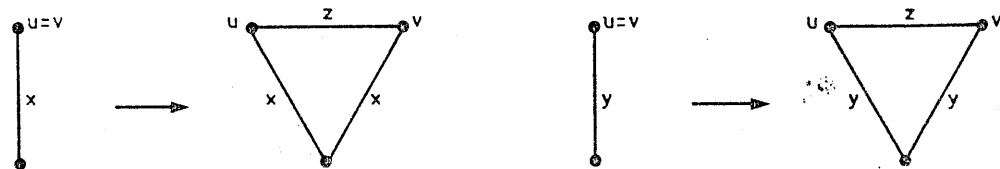


FIGURE 3.18 Assignment of edge colours to $T(n+1)$.

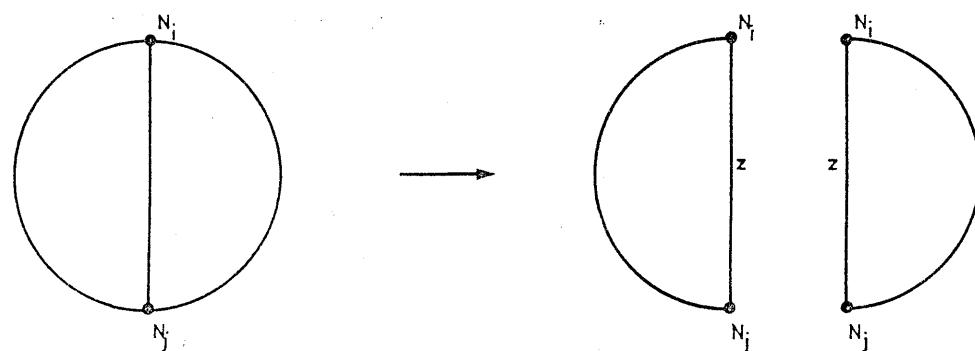


FIGURE 3.19 Decomposition of $T(p,q)$.

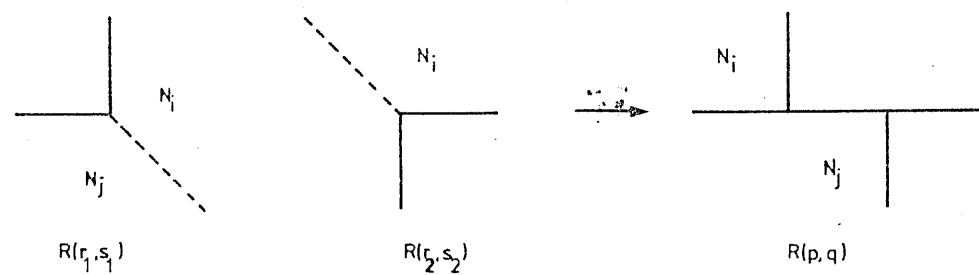


FIGURE 3.20 Assembly of $R(p,q)$ from $R(r_1, s_1)$ and $R(r_2, s_2)$.

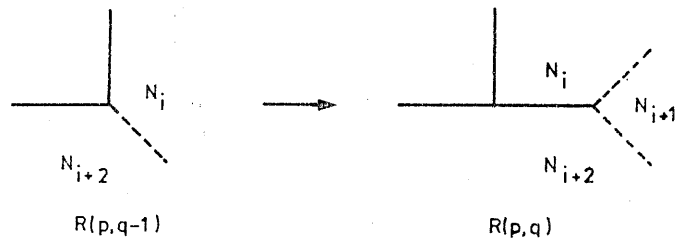


FIGURE 3.21 Construction of $R(p, q)$ from $R(p, q-1)$.

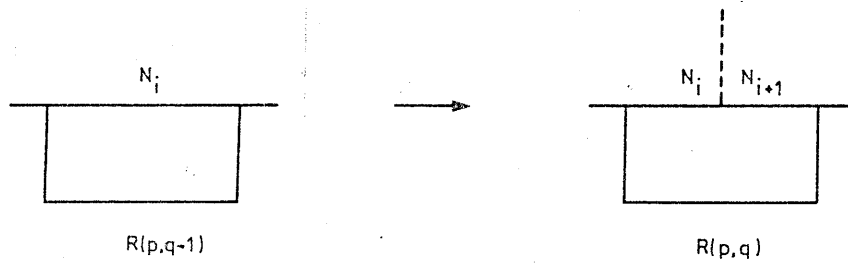


FIGURE 3.22 Construction of $R(p, q)$ from $R(p, q-1)$.

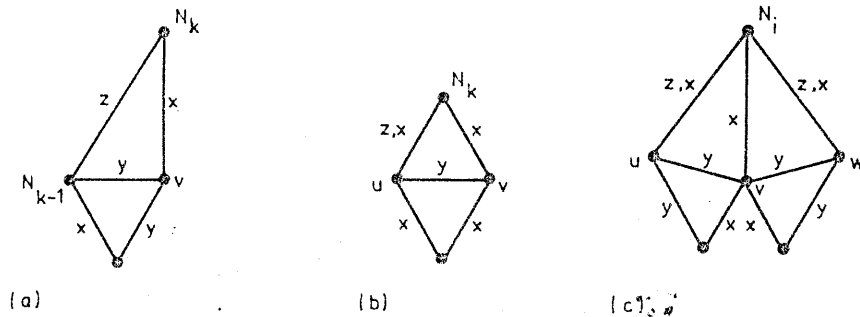


FIGURE 3.23 Configurations in $T(p, q)$.

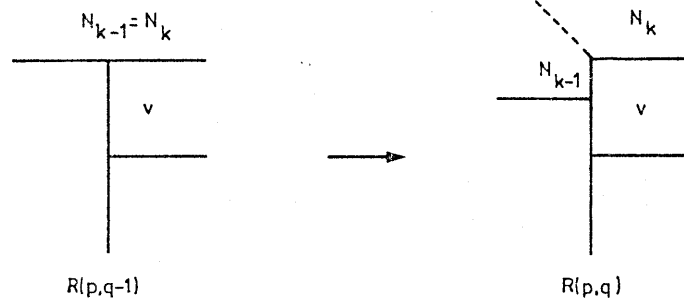


FIGURE 3.24 Construction of $R(p, q)$ from $R(p, q-1)$.

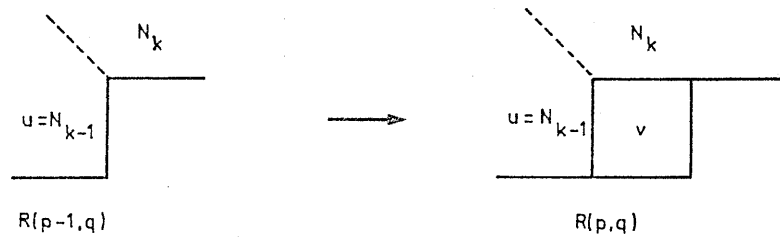


FIGURE 3.25 Construction of $R(p, q)$ from $R(p-1, q)$.

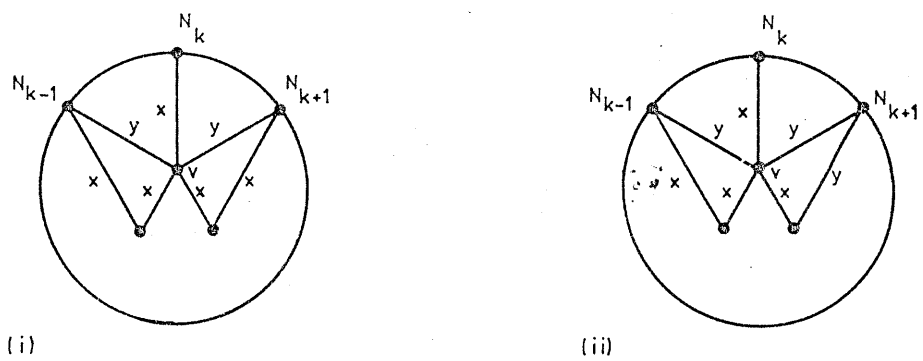


FIGURE 3.26 Configurations in $T(p, q)$.

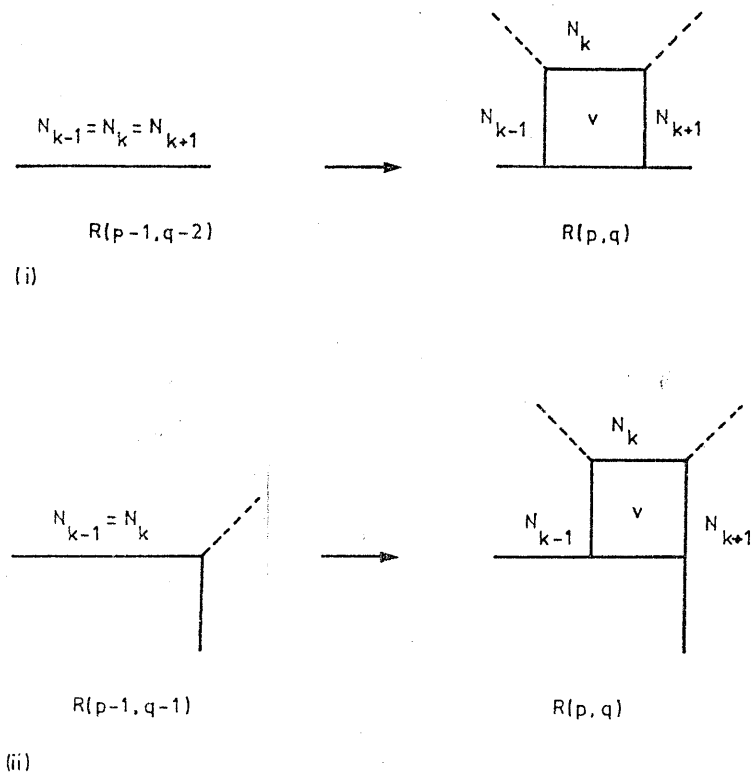


FIGURE 3.27 Construction of $R(p, q)$ from (i) $R(p-1, q-2)$ and (ii) $R(p-1, q-1)$.

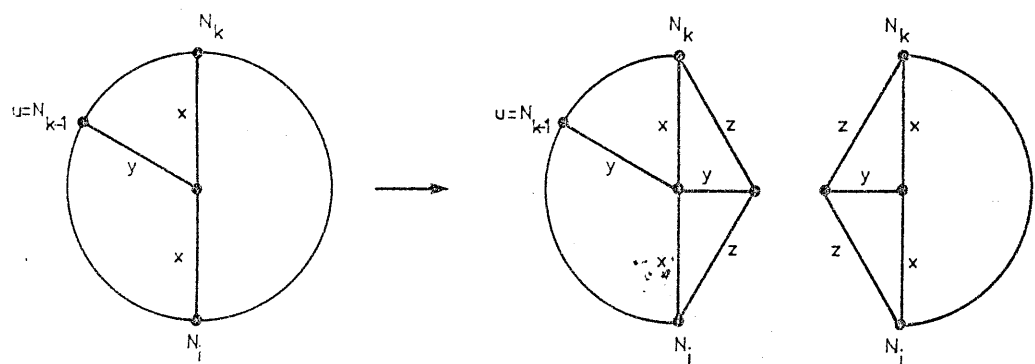


FIGURE 3.28 Decomposition of $T(p, q)$.

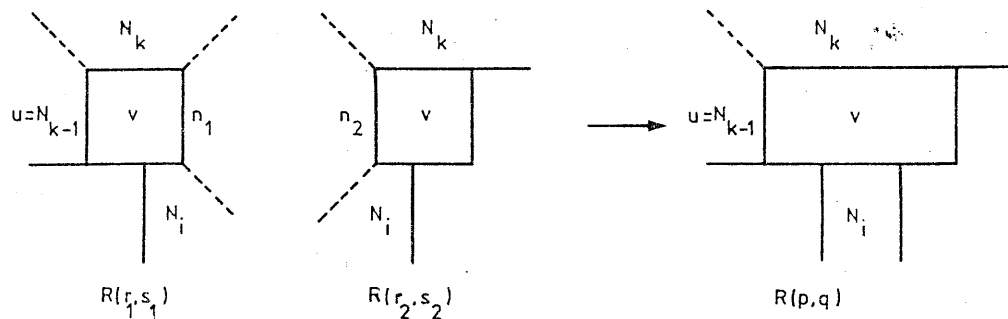


FIGURE 3.29 Assembly of $R(p, q)$ from $R(r_1, s_1)$ and $R(r_2, s_2)$.

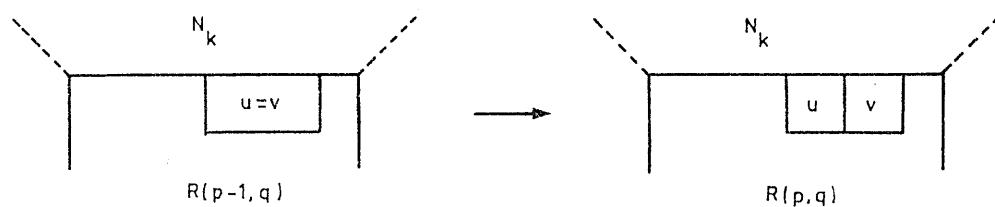


FIGURE 3.30 Construction of $R(p, q)$ from $R(p-1, q)$.

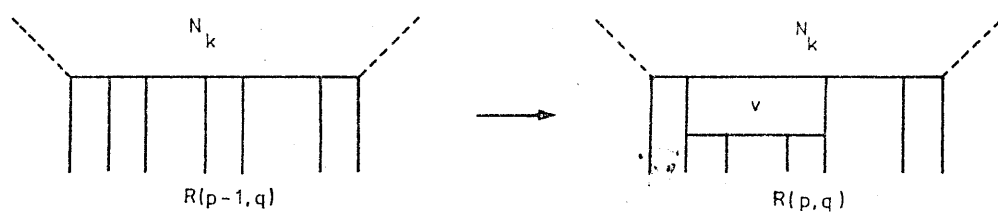


FIGURE 3.31 Construction of $R(p, q)$ from $R(p-1, q)$.

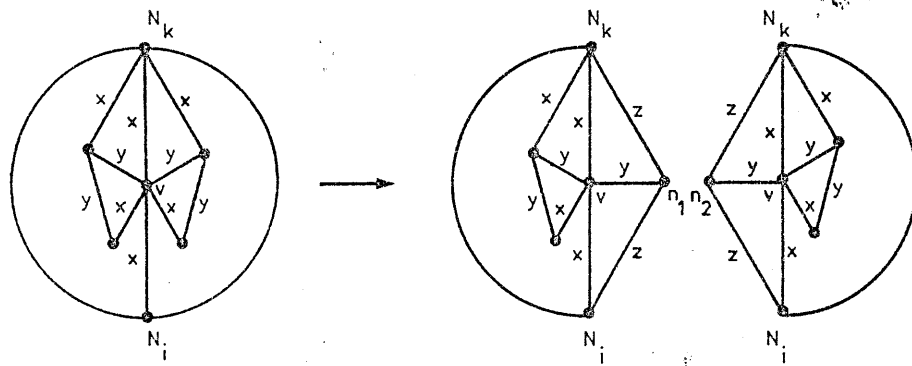


FIGURE 3.32 Decomposition of $T(p,q)$.

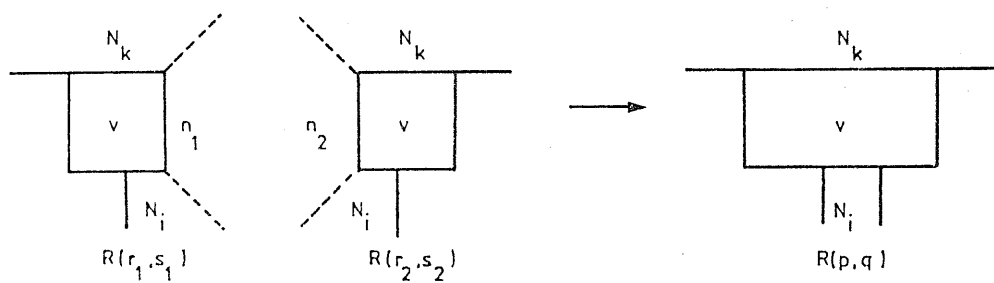


FIGURE 3.33 Assembly of $R(p,q)$ from $R(r_1, s_1)$ and $R(r_2, s_2)$.



FIGURE 3.34 The $[0,4]$ rectangular map and its dual.

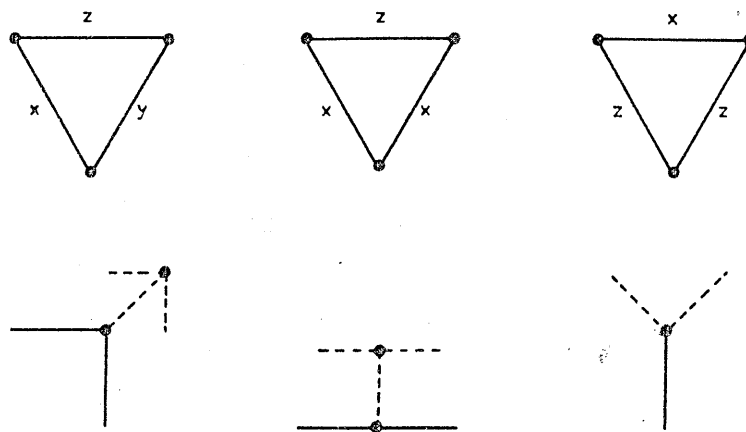


FIGURE 3.35 Corners in rectangular maps.

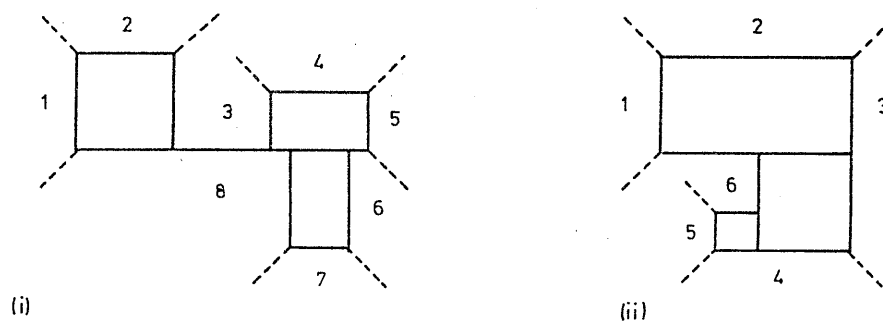


FIGURE 3.36 (i) a $[3,8]$ rectangulation and (ii) a strong $[3,6]$ rectangulation.



FIGURE 3.37 The "top left corner" rectangle in a rectangulation.

n	4	5	6	7	8	9	10	11
	1	3	13	68	399	2536	16 965	118 668

TABLE 3.1 3-connected trivalent rooted plane maps with $n \leq 11$ faces.

n	3	4	5	6	7	8	9	10	11	12
	1	1	1	2	5	14	50	233	1249	7595

TABLE 3.2 3-connected trivalent planar maps with $n \leq 12$ faces.

n	4	5	6	7	8	9
	4	24	176	1344	13 056	124 032

TABLE 3.3 2-connected trivalent rooted plane maps with $n \leq 9$ faces.

r/s	3	4	5	6	7	8	9	10
0	1	2	5	14	42	132	429	1430
1	1	5	21	84	330	1287	5005	
2	3	20	105	504	2310	10 296		
3	13	100	595	3192	16 170			
4	68	570	3675	21 252				
5	399	3542	24 150					
6	2530	23 400						
7	16 965							

TABLE 3.4 Rooted $[r,s]$ triangulations, $r + s \leq 10$.

r/s	3	4	5	6	7	8	8	10
0	1	1	1	4	6	19	49	150
1	1	2	5	16	48	164	599	
2	1	6	21	88	330	1302		
3	5	26	119	538	2310			
4	24	147	735	3568				
5	133	892	4830					
6	846	5876						
7	5661							

TABLE 3.5 $[r,s]$ triangulations, $r + s \leq 10$, up to orientation preserving boundary homeomorphisms.

r/s	3	4	5	6	7	8	9	10
0	1	1	1	3	4	12	27	82
1	1	2	4	11	28	91	311	
2	1	5	14	53	178	685		
3	4	18	69	295	1196			
4	16	88	396	1867				
5	78	489	2503					
6	457	3071						
7	2938							

TABLE 3.6 $[r,s]$ triangulations, $r + s \leq 10$, up to boundary homeomorphisms.

r/s	3	4	5	6	7	8	9	10
0	1	1	1	2	2	5	5	14
1	1	2	3	6	8	18	23	
2	1	4	7	18	26	68		
3	3	10	19	52	82			
4	8	29	57	166				
5	23	86	176					
6	68	266						
7	215							

TABLE 3.7 $[r,s]$ triangulations, $r + s \leq 10$, which have reflection symmetry.

r/s	3	4	5	6	7	8	9	10
0	1	2	5	14	42	132	429	1430
1	1	1	6	28	120	495	2002	
2	0	2	15	84	420	1980		
3	1	6	50	308	1680			
4	3	22	195	1274				
5	12	91	840					
6	52	408						
7	241							

TABLE 3.8 Rooted simple $[r,s]$ triangulations, $r + s \leq 10$.

r	1	2	3	4	5	6	7	8	9	10
	1	2	6	24	116	642	3938	26 194	186 042	1395 008

TABLE 3.9 Validly coloured rooted simple $[r,4]$ triangulations, $r \leq 10$.

r	1	2	3	4	5	6	7	8	9	10
	1	2	6	22	91	408	1938	9614	49 335	260 130

TABLE 3.10 Rooted simple $[r,4]$ triangulations, $r \leq 10$.

4.1 Introduction

In section 3.4 rectangulations were considered as a transformation, by the addition of a colour (or label) structure, of certain classes of trivalent plane maps. The colour structure specified the directions of the edges in the rectangulation. In a sense edges were aligned so that sets of edges formed lines. However, if rectangulations are considered such that the component lines lie on the lines of a grid, then roughly speaking there are more detailed alignment relations arising from the ordering imposed by the grid structure. Thus for a given colour structure on a trivalent plane map there may be many rectangulations considered in this new sense as arrangements of lines on the lines of a grid. For example the arrangements in figure 4.1 correspond to the same coloured simple [6,4] triangulation.

This chapter will examine the representation, classification and construction of architectural plans whose walls lie along a specified number of directions. In the case of rectangulations there are two directions at right angles. The walls are represented by line segments and adjacent walls in the same direction form maximal walls, which, in turn, are represented by maximal line segments. This examination is restricted to the case in which there are two specified directions at right angles.

A finite set of closed, finite, line segments (or lines for brevity) in two dimensional euclidean space is called a shape. Any given shape can be considered as a unique finite set of maximal lines,

that is, a set of lines such that the union of any two is not a line (Stiny 1980). Denote a shape by its set of maximal lines S . A subshape of a shape is a shape, each of whose maximal lines is contained in a maximal line of the original shape. Consider a subshape S' of a shape S such that $S' \subseteq S$ and no line in S' is coincident with a line in $S - S'$. Then S' is called a component of S and S is connected if it has one component.

In this section the architectural plans are considered whose walls are represented by lines in precisely two perpendicular directions. The "grid structure" mentioned earlier is referred to as a rectangular grating, or simply a grating. A rectangular grating is a shape in which the maximal lines are parallel to cartesian coordinate axes, four of the maximal lines share endpoints (the boundary rectangle) and the rest have endpoints coincident with the boundary rectangle (the internal maximal lines). A grating is an (ℓ, m) grating if there are $\ell + 1$ maximal lines parallel to the y-axis and $m + 1$ maximal lines parallel to the x-axis.

A subshape of an (ℓ, m) rectangular grating such that each maximal line has endpoints at a point of intersection of the maximal lines in the original grating is an (ℓ, m) rectangular shape. An (ℓ, m) rectangular shape is minimal if it is not an (ℓ', m') rectangular shape with $\ell' + m' < \ell + m$.

Each (ℓ, m) grating can be mapped onto the corresponding (ℓ, m) unit square grid and this mapping induces a map of the (ℓ, m) rectangular shapes onto rectangular shapes on an (ℓ, m) square grid. Since interest centres upon alignment relations rather than precise geometric dimensions of the grating, these rectangular shapes may

be considered as representative or canonical rectangular shapes. Rectangular shapes are equivalent if the corresponding representative shapes are the same under translation. Other equivalences can also be defined on rectangular shapes according to the euclidean transformations on the canonical shapes. The inverse of the map which assigns the canonical rectangular shape to a given rectangular shape may be considered as an operation which dimensions the representative shape.

4.2 Classification of rectangular shapes

Some classes of rectangular shapes have been examined as representations of architectural arrangements on grids. These are usually minimal canonical shapes. Rectangular dissections have been examined (Mitchell et al 1976, Earl 1977, Krishnamurti and Roe 1978, Bloch 1979) together with polyomino shapes (March and Matela 1974). It ought to be noted that the rectangular dissections which are considered for the catalogue in Mitchell et al (1976) seem to be precisely the rectangulations considered in section 3.4 above. In this thesis the term rectangular dissection is reserved for arrangements which are specifically embedded on a grating or grid. The terms rectangular dissection and rectangulation have been used interchangeably in the previous literature.

Definitions of rectangular dissections and general polyomino-type shapes are given first and then other classes of rectangular shapes defined. A rectangular dissection is a rectangular shape in which four maximal lines share endpoints (the boundary rectangle) and the rest have endpoints coincident with other maximal lines but not at their endpoints (Earl 1978).

A polyomino is defined completely by its boundary polygons. These boundary polygons are considered to form the rectangular shape. A general polyomino-type shape is a rectangular shape in which each maximal line shares both endpoints with other maximal lines. On an (ℓ, m) square grid these shapes can be "filled in" with the grid lines to produce arrangements of squares. The result is not necessarily a polyomino but a general arrangement of squares.

An endpoint of maximal lines in a rectangular shape may be

- (a) not coincident with any other maximal line in the shape
- (b) coincident with another maximal line at its endpoint
- (c) coincident with another maximal line not at its endpoint.

There are seven types of rectangular shape, based upon the above distinction, depending on whether they only have endpoints of kinds $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$. For instance a rectangular dissection is a rectangular shape of type $\{b,c\}$ with precisely four maximal lines which share endpoints. There are no rectangular shapes of type $\{c\}$ and rectangular shapes of type $\{b\}$ are the general polyomino-type shapes. A rectangular shape is called trivalent if any two maximal lines either do not intersect or intersect at an endpoint of at least one of the maximal lines.

4.3 Construction of rectangular shapes

To examine the construction of these classes of shapes use is made of subshape replacement rules. As for the construction of maps in Chapters 2 and 3 and the construction of graphs in the later Chapter 6. These replacement rules are given in an informal way. A rigorous and formal approach would require the specification of map and graph grammars. In this chapter subshape replacement

rules are presented informally as schemata and it is relatively straightforward to formalize these schemata as parametric shape grammars in the sense of Stiny (1977). A further point should perhaps be made. The grammars mentioned above usually have replacement rules in which the submaps, subgraphs and subshapes are labelled or marked. Concentrating on shapes, the initial shape is labelled or marked at the beginning; the labels and markers are erased to terminate a construction using the grammar. If all the labels and markers are erased then a shape in the language specified by the grammar is constructed. By choice of erasing rules it is possible to test for specified properties in the shape being constructed.

The (ℓ, m) rectangular shapes of type $\{a, b, c\}$ are trivially constructed from an (ℓ, m) grating, marked as shown in figure 4.2(i) using the shape and erasing rules in figure 4.2(ii). For (ℓ, m) rectangular shapes the initial shape is taken as an (ℓ, m) grating. The first rule marks this shape as shown in figure 4.3. Consider the rules shown in figure 4.4(i) which remove lines and change the markers to produce marked subshapes of the original marked grating. Figure 4.4(ii) gives erasing rules for the markers.

The rectangular shapes of the different types are generated by subsets of these rules which consist of rules (1)-(6) together with the erasing rules given below.

$\{a, b, c\}$	\longleftrightarrow	$\{A, B, C, D, E, F\}$
$\{b, c\}$	\longleftrightarrow	$\{A, C, D, E, F\}$
$\{a, c\}$	\longleftrightarrow	$\{A, B, D, E, F\}$
$\{a, b\}$	\longleftrightarrow	$\{A, C, D, F\}$
$\{b\}$	\longleftrightarrow	$\{A, C, D, F\}$
$\{a\}$	\longleftrightarrow	$\{A, B, D, F\}$

Trivalent shapes of the corresponding classes are generated by deleting rule A from the above subsets of rules.

Rectangular dissections are a subset of the rectangular shapes of type {b,c}. Consider the marking of the initial grating shown in figure 4.5 which ensures that the boundary rectangle is preserved during a generation which uses only the rules 1,2 and 3. Rectangular dissections are generated by the rules {1,2,3, A,D,E} and trivalent rectangular dissections by {1,2,3, D,E}. Minimal rectangular dissections are generated by {1,2, A,D,E} and minimal trivalent dissections by {1,2, D,E}.

A class of rectangular dissections which has been investigated (Earl, 1977, 1978) consists of non-aligned minimal dissections. These possess exactly one maximal line on each grid line. They are also defined as those dissections produced by the rules {2, A,D,E}. The minimal trivalent non-aligned dissections are generated by the rules {2, D,E}.

An extensive algorithmic investigation of rectangular dissections is given in Krishnamurti and Roe (1978) and enumeration results in Bloch and Krishnamurti (1978). The methods used are not immediately applicable to general rectangular shapes since the rectangular "spaces" in the dissections are used as an essential coding feature. Lines appear as boundaries between the spaces.

Further classification of rectangular shapes is given by considering the intersections of maximal lines. An intersection may be

(d) At the endpoint of no maximal line,

- (e) At the endpoint of precisely one maximal line,
- (f) At the endpoint of precisely two maximal lines.

A maximal line may intersect

- (g) No other maximal lines,
- (h) At least one other maximal line,
- (i) Precisely one other maximal line.

These distinctions give rise to 6^3 types depending on whether the intersections and maximal lines are only of particular kinds corresponding to non-empty subsets of $\{d, e, f, g, h, i\}$. For example, the types of rectangular shape corresponding to $\{d, i\}$, $\{e, i\}$, $\{f, i\}$ consist of $+$, \top and \perp configurations respectively (figure 4.6). These classes can be generated using the rules (1)-(6) and further erasing rules (figure 4.7).

4.4 Representations of rectangular shapes

The aim of this section is to examine the incidence structures among the elements, that is, the regions and maximal lines, in trivalent rectangular shapes. First the maximal line adjacency structures are considered and then the incidence structures of regions and maximal lines.

4.4.1 Maximal line adjacency

Suppose that the maximal lines in a trivalent rectangular shape are represented by vertices and the intersections of maximal lines by edges between the corresponding vertices. The cyclic order of the edges at a given vertex is the cyclic order of the intersections of maximal lines with the corresponding maximal line. The result is a plane pseudo-map called the maximal line adjacency map, whose faces represent the regions in the corresponding trivalent

rectangular shape. The distinguished face represents the external region (figure 4.8).

The maximal line adjacency map of a connected trivalent rectangular shape is a plane map in which each face has even degree and there are no 2-cycles. Let such a map be called a q-map. Two q-maps are considered equivalent if there is an orientation preserving boundary homeomorphism between them. Given a connected trivalent rectangular shape all rotated versions have the same maximal line adjacency map.

The maximal line adjacency map of a connected trivalent rectangular shape of type $\{b,c\}$ is a q-map with no vertices of degree one. The maximal line adjacency map of a trivalent rectangular dissection is a q-map with all faces of degree four.

A q-map, Q , has all faces of degree four if and only if

$$|V(Q)| = |F(Q)| + 2. \quad \text{Let a generalized rectangular dissection}$$

be a connected trivalent rectangular shape of type $\{b,c\}$ in which each finite region is bounded by a rectangle. The maximal line adjacency map of a generalized trivalent rectangular dissection is a q-map with no vertices of degree one and all non-distinguished faces of degree four.

Proposition 4.1. Each q-map is the maximal line adjacency map of a connected trivalent rectangular shape.

Proof. The proof is by induction on the number of vertices.

Suppose the proposition holds for all q-maps with at most n vertices.

Let $Q(n+1)$ be a q-map with $n+1$ vertices. If $Q(n+1)$ has a 1-vertex separator then it is obtained from q-maps with less than $n+1$ vertices.

by identifying vertices. These q-maps are the maximal line adjacency maps of connected trivalent rectangular shapes, which can be assembled at the maximal line corresponding to the 1-vertex separator to form a connected trivalent rectangular shape whose maximal line adjacency map is $Q(n+1)$.

Now suppose that $Q(n+1)$ is 2-vertex connected and consider a non-distinguished face with (cyclically ordered) vertices $\{u_1, u_2, \dots, u_{2k}\}$ in its boundary. If $k > 2$ then at least one of the pairs of vertices $\{u_1, u_3\}$ or $\{u_2, u_{2k}\}$ does not belong to a 4-cycle since $Q(n+1)$ has no cycle of odd length. Suppose $\{u_2, u_{2k}\}$, and identify u_2 and u_{2k} forming a q-map with n vertices (figure 4.9(i)), which by induction corresponds to a connected trivalent rectangular shape. If $k = 2$ then at least one of the pairs of vertices $\{u_1, u_3\}$ or $\{u_2, u_4\}$ does not belong to a 4-cycle other than (u_1, u_2, u_3, u_4) . Suppose $\{u_2, u_4\}$, and identify u_2 and u_4 , forming a q-map with n vertices (figure 4.9(ii)) which by induction corresponds to a connected trivalent rectangular shape. One of the operations of the types shown in figures 4.9 (iii) and (iv), for the cases $k = 2$ and $k > 2$, respectively, yields a connected trivalent rectangular shape whose maximal line adjacency map is $Q(n+1)$. To complete the induction, the edge map is the maximal line adjacency map of a connected trivalent rectangular shape.

To examine the enumeration of q-maps, the rooted q-maps are considered. Note first that a q-map has a unique 2-vertex colouring which corresponds to the two sets of maximal lines in a corresponding connected trivalent rectangular shape. Consider a

rooted q -map, Q . If the root vertex is coloured x , then a 2-vertex colouring with x and y , say, is uniquely determined. Consider the rooted map constructed from the $(W(F), W(V))$, rooted derived map of Q , by deleting $(W(F), W(E))$ edges and contracting each $(W(E), y)$ edge. Let the $W(F)$ vertices be labelled z . Let the result be called a rooted z -map. Each rooted z -map is a 3-coloured rooted triangular map with root edge from a z -coloured to an x -coloured vertex. However, such a map may have multiple (x, z) , (y, z) and (x, y) edges. It is a rooted z -map if and only if it has no multiple (x, y) edges. Consider the operation shown in figure 4.10 which splits an (x, y) edge and inserts a 3-coloured rooted triangular map. Each 3-coloured rooted triangular map with root edge directed from a z -coloured to an x -coloured vertex is constructed from a unique rooted z -map by a unique set of these operations at distinct (x, y) edges in the original rooted z -map. If the original rooted z -map has n_1 z -vertices of degree $2i \geq 4$ then there are $\prod n_i$ distinct (x, y) edges at which the operation may be applied.

The 3-coloured rooted triangular maps with root edge directed from a z -coloured vertex to an x -coloured vertex, with n_i non-root z -vertices of degree $2i$, $i = 1, 2, \dots$, and root vertex degree $2k$, $k = 1, 2, \dots$, have been enumerated in their dual form by Tutte (1963). Let the number with n_i non-root z -vertices of degree $2i \geq 4$ and root vertex degree $2k \geq 4$ be denoted by $a(k; n_2, n_3, \dots)$. Let $b(k; n_2, n_3, \dots)$ denote the number of rooted z -maps with n_i non-root z -vertices of degree $2i \geq 4$ and root vertex degree $2k \geq 4$. Let $A(x_1, x_2, x_3, \dots) = A(\underline{x})$ and $B(x_1, x_2, x_3, \dots) = B(\underline{x})$ be the corresponding generating functions. By using the construction of 3-coloured rooted triangular maps given above these generating

functions are related in the following way:

$$A(\underline{x}) = B(x_1\{1+A(\underline{x})\}, x_2\{1+A(\underline{x})\}^2, x_3\{1+A(\underline{x})\}^3, \dots). \quad (1)$$

Let $a(k;n)$ and $b(k;n)$ denote the number of 3-coloured rooted triangular maps and rooted z -maps respectively, with n non-root z -vertices each of degree four and root vertex degree $2k \geq 4$. If the corresponding generating functions are $A(x,y)$ and $B(x,y)$ then

$$A(x,y) = B(x\{1+A(x,y)\}, y\{1+A(x,y)\}^2). \quad (2)$$

Let $a(n)$ and $b(n)$ denote the number of 3-coloured rooted triangular maps and rooted z -maps respectively, with n z -vertices each degree four. If the corresponding generating functions are $A(x)$ and $B(x)$ then

$$A(x) = B(x\{1+A(x)\}^2). \quad (3)$$

Finally let $c(k;m,n)$ and $d(k;m,n)$ denote the number of 3-coloured rooted triangular maps and rooted z -maps respectively, with m x and y -vertices, n non-root z -vertices each of degree at least four and root vertex degree $2k \geq 4$, then

$$c(k;m,n) = \sum a(k;n_2, n_3, \dots), \quad (4)$$

where the summation is taken over n_2, n_3, \dots , which satisfy $n_2 + n_3 + \dots = n$ and $k + n_2 + 2n_3 + \dots = m - 1$. If the corresponding generating functions are $C(x_1, x_2, x_3) = C(\underline{x})$ and $D(x_1, x_2, x_3)$ then

$$C(\underline{x}) = \frac{1}{1+C(\underline{x})} D(x_1, x_2\{1+C(\underline{x})\}, x_3\{1+C(\underline{x})\}). \quad (5)$$

These generating functions are now interpreted for rooted q -maps. The $b(k; n_2, n_3, \dots)$ are the numbers of rooted q -maps with distinguished face degree $2k$ and n_i non-distinguished faces of degree $2i$. The $b(k;n)$ are the numbers of rooted q -maps with

distinguished face degree $2k$ and n non-distinguished faces each of degree four. The $d(k;m,n)$ denote the number of rooted q -maps with distinguished face degree $2k$, m vertices and n non-distinguished faces. The $b(n)$ denote the number of rooted q -maps with n faces each of degree four.

Consider a q -map with all faces degree four. This is the maximal line adjacency map of a trivalent rectangular dissection. The vertices in the boundary of the distinguished face correspond to the maximal lines forming the boundary of the dissection. Consider the rooted maximal line adjacency map with root edge from the vertex corresponding to the "top" line in the boundary to the "left" line in the boundary. These rooted maximal line adjacency maps of trivalent rectangular dissections with $n-1$ internal regions are enumerated by $b(n)$. They are the rooted quadrangulations of a quadrilateral (Brown 1965) with $n-2$ internal vertices and thus

$$b(n) = \frac{12(3n-4)!}{(n-2)!(2n)!} = \frac{2(3n-3)!}{n!(2n-1)!} \quad (6)$$

The corresponding rooted z -maps are the $(W(E), W(V))$ or $(W(E), W(F))$ rooted derived maps of the 2-vertex connected rooted plane maps. Thus $b(n)$ is the number of 2-vertex connected rooted plane maps with n edges (see section 2.2.2). These rooted plane maps provide further representations of trivalent rectangular dissections; the network representations. To each rooted q -map there are two network representations, the x and y -networks, depending on whether the $(W(E), W(V))$ or $(W(E), W(F))$ rooted derived maps are considered. The x and y -networks are dual rooted plane maps. The x -network has vertices corresponding to "horizontal" maximal lines, faces corresponding to "vertical" maximal lines and

edges corresponding to regions. The y -network has vertices corresponding to "vertical" maximal lines, faces to "horizontal" maximal lines and edges to the regions (figure 4.11).

The q -maps with all faces of degree four are the quadrangulations of a quadrilateral up to orientation preserving boundary homeomorphisms. They are enumerated by Brown (1965), although there is an error in the generating function equation for the rooted quadrangulations of type " $[n,m;2]$ " since equation (11.2) requires another term, namely $(1 + y^4 \overline{W})y^4$, to be added on the right hand side. Taking this into account the number of quadrangulations of a quadrilateral with n internal vertices, up to orientation preserving boundary homeomorphisms, that is, the number of maximal line adjacency maps of trivalent rectangular dissections with $n+1$ internal regions, is given by:

$$\frac{1}{4} ({}_1U_n + 2{}_2U_n + 2{}_4U_n), \quad (7)$$

$$\text{where } {}_1U_n = \frac{12 (3n + 2)!}{n! (2n + 4)!},$$

$$\begin{aligned} {}_2U_{2n} = & \frac{(3n)!}{n! (2n + 1)!} + \frac{3(3n - 1)!}{(n - 1)! (2n + 1)!} + \frac{3(3n - 4)!}{(n - 2)! (2n - 1)!} \\ & + \frac{5(3n - 5)!}{(n - 3)! (2n - 1)!}, \end{aligned}$$

$${}_2U_{2n+1} = \frac{6 (3n + 2)!}{n! (2n + 3)!},$$

$${}_4U_{4n} = \frac{(3n)!}{n! (2n + 1)!},$$

$${}_4U_{4n+k} = 0 \text{ if } k = 1, 2, 3.$$

Consider the class of trivalent rectangular dissections which have no subshape, except a single rectangle, which is a trivalent rectangular dissection. They are irreducible dissections, in the sense that they are not formed by the "addition" of two or more dissections. The maximal line adjacency maps of irreducible trivalent rectangular dissections are precisely the q -maps with all faces degree four and with each 4-cycle bounding a face. These are the simple quadrangulations of a quadrilateral (Mullin and Schellenberg 1968). The "addition" of dissections, considered in terms of the q -maps, consists in the quadrangulation of quadrangular faces, including the external face (figure 4.12).

The rooted q -maps with all faces degree four and with each 4-cycle bounding a face are enumerated by Mullin and Schellenberg (1968). The corresponding rooted z -maps are the $(W(E), W(V))$ or $(W(E), W(F))$ rooted derived maps of the 3-vertex connected rooted plane maps. Thus the network representations of the irreducible trivalent rectangular dissections are precisely the 3-vertex connected rooted plane maps. The "addition" of dissections, considered in terms of the network representation, consists in a simple operation, at any edge, including the root-edge (figure 4.13).

4.4.2 Region-maximal line incidence

Consider a q -map, Q , which is the maximal line adjacency map of a given connected trivalent rectangular shape of type $\{b, c\}$. Suppose that the vertices corresponding to maximal lines parallel to the x and y -axes are labelled x and y respectively. The labels form the 2-colouring of Q . Consider the derived map Q' and the sub-map of Q' obtained by deleting vertices corresponding to the

edges of Q , and labelling as z the vertices corresponding to the faces of Q . There is a distinguished z -vertex, denoted by z^* , say, corresponding to the distinguished face of Q . This map is called the region-maximal line incidence map of the shape (figure 4.14(ii)). A map constructed in this way with x, y and z labels on the vertices is called an s -map. Two s -maps are considered equivalent if there is an orientation preserving homeomorphism between them which preserves x, y and z labels together with the distinguished z -vertex.

The s -maps are the 2-vertex connected planar maps with all faces degree four, vertices labelled x, y, z such that each face has two non-adjacent z -vertices, one x -vertex and one y -vertex. No two faces share both an x and a y -vertex. There is also a distinguished z -vertex.

Each s -map is the region-maximal line incidence map of a connected trivalent rectangular shape of type $\{b, c\}$. If X, Y, Z denote the sets of x, y, z -vertices, respectively, in an s -map and there are n_i non-distinguished z -vertices of degree $2i \geq 4$, and z^* -vertex degree $2k \geq 4$, then since there are $\sum n_i + k$ faces, Euler's formula gives $|X| + |Y| = \sum (i-1)n_i + k + 1$. If all non-distinguished z -vertices are degree four then $|X| + |Y| = |Z| + k$, and if all z -vertices are degree four then $|X| + |Y| = |Z| + 2$. Also if $|X| + |Y| = |Z| + 2$ then all z vertices are degree four. Thus an s -map is the region-maximal line incidence map of a trivalent rectangular dissection if and only if $|X| + |Y| = |Z| + 2$.

4.4.3 Oriented region-maximal line incidence

If the edges of a region-maximal line incidence map of a given connected trivalent rectangular shape of type $\{b, c\}$ are directed

from x to z (y to z) if the x -vertex (y -vertex) represents a maximal line above (to the left of) the region represented by the z -vertex and from z to x (z to y) if the x -vertex (y -vertex) represents a maximal line below (to the right of) the region represented by the z -vertex. The result is called the oriented regional-maximal line incidence map (figure 4.14(iii)).

Consider an s -map with directions on its edges. It has an associated labelling according to the scheme in figure 4.15 (up to reversal of directions). Such an oriented s -map is validly oriented if the associated labelling has precisely three "1" labels in each face, the sum of the labels at each non-distinguished z -vertex is $+4$, and at z^* is -4 , the sums of the labels at each x and y -vertex is $+2$.

There are four possibilities for the labels in the faces (figure 4.16), thus eight possibilities for the directions on the edges in a face. The oriented regional-maximal line incidence map of a connected trivalent rectangular shape of type $\{b,c\}$ is a validly oriented s -map. The eight types of face correspond to the possible intersections of maximal lines (figure 4.17).

Two validly oriented s -maps are considered equivalent if the underlying s -maps are equivalent under an orientation preserving homeomorphism which also preserves the directions on the edges.

Before proceeding a result concerned with cycles (not necessarily directed) in a validly oriented s -map is demonstrated. A cycle C , length $2k \geq 4$ divides the labels at each vertex on the cycle into two sets, the interior and exterior, say. Let there be

m interior "-1" labels and n interior "+1" labels at z-vertices in C. Also let there be p interior "+1" labels at x and y-vertices on C. Let there be v_i z-vertices in the interior of C, with degree $2i \geq 4$, and $\sum v_i = v$. Also let there be w x and y-vertices in the interior of C. Suppose first that the distinguished z-vertex either lies on C or in the exterior of C. The number of faces in the interior of C is

$$\sum iv_i + 1/2 (m + n). \quad (8)$$

The number of "-1" labels in the interior of C is

$$\sum (i - 2)v_i + m. \quad (9)$$

Thus the number of faces in the interior of C is also

$$2w + p - (\sum (i - 2)v_i + m). \quad (10)$$

Equations (8) and (10) give

$$2w + 2v + p - 2 \sum iv_i - 3/2 m - 1/2 n = 0. \quad (11)$$

Euler's formula gives

$$v + w + 2k - (\sum iv_i + 1/2 (m + n)) - k = 1. \quad (12)$$

Equations (11) and (12) give

$$4k = 2p + (n - m) + 4. \quad (13)$$

If $\ell = n - m$ is the sum of the labels at z-vertices on C then

$$4k = 2p + \ell + 4. \quad (14)$$

If the distinguished z-vertex lies in the interior of C then it may be shown that,

$$4k = 2p + \ell - 4. \quad (15)$$

Equations (14) and (15) can be used to give results on directed paths and cycles in a validly oriented s-map S , with all non-distinguished z -vertices of degree four. Let directed paths consisting of only (x,z) and (y,z) edges be called (x,z) and (y,z) directed paths, respectively. It is straightforward to show that each directed cycle contains the distinguished z -vertex, z^* . Thus if z^* is removed from S the result is an acyclic directed map.

It is shown that each pair of non-distinguished z -vertices, z_1, z_2 in S belong to an (x,z) or (y,z) directed path, which avoids z^* . Both z_1 and z_2 lie on directed (x,z) and (y,z) cycles, which pass through z^* . The configuration in figure 4.18 occurs. If $z_3 = z_2$ or $z_4 = z_1$ the result follows. Suppose $z_3 \neq z_2$ and $z_4 \neq z_1$, let C denote the boundary cycle and $Z(C)$ denote the z -vertices on C and in the interior. Note that $z^* \notin Z(C)$. Consider the vertices x_1 and y_1 adjacent to z_3 . There is a directed edge from x_1 to z_5 or from y_1 to z_5 for some $z_5 \in Z(C) - z_3$, suppose (x_1, z_5) . Thus there is a directed (x,z) path from x_1 to a vertex on C . Either the required directed path is constructed or a similar configuration to figure 4.18 is created with bounding cycle C' , $Z(C') \subset Z(C)$. Repeating the argument eventually yields the required directed path.

Finally each pair of non-distinguished z -vertices in S do not belong to both (x,z) and (y,z) directed paths which avoid z^* . This follows immediately from (14) and (15) (see also Flemming (1980)).

Consider a valid orientation of an s-map, S . Note that each s-map has a valid orientation since each s-map is the region-maximal line incidence map of a connected trivalent rectangular shape of type $\{b,c\}$. Reversing the directions on all edges in S

yields another valid orientation. Suppose that a subset of edges in S , forming a connected submap of S , have their directions reversed such that the result is another valid orientation of S . Such a submap may be considered as the interior edges in a connected region whose boundary is formed by edges of S . Consider a $2k$ -cycle C in this boundary. The interior labels at vertices on C are of the form shown in figure 4.19. If the sum of the interior labels at z -vertices on C is ℓ , then, since the sum of interior labels at x and y -vertices on C is k , equation (14) gives $2k = \ell + 4$, if the distinguished z -vertex lies on or outside C and equation (15) gives $2k = \ell - 4$ if the distinguished z -vertex lies inside C . Let a submap in a validly oriented s -map whose edge directions may be reversed be called a reversible submap. Any valid orientation of S can be obtained by reversing all edge directions or the edge directions in reversible submaps in a given valid orientation.

Suppose that S has all z -vertices of degree four, that is, it is the region-maximal line incidence map of a trivalent rectangular dissection. Consider a valid orientation of S and a $2k$ -cycle, C , in the boundary of a reversible submap. If the distinguished z -vertex lies on C then $\ell = k - 2$ and $\ell = k$ otherwise. If the former then $k = 2$, but no two faces of an s -map share both an x and a y -vertex, thus this is impossible. Thus $\ell = k$ and $k = 4$. The internal edges of a configuration of the type shown in figure 4.20 form the only type of reversible submap in S .

Consider an s -map S with all z -vertices degree four. It is the region-maximal line incidence map of a trivalent rectangular dissection D . Consider the corresponding valid orientation of S .

Any reversible submap corresponds to a "pinwheel" configuration of the type shown in figure 4.21 where D' denotes a trivalent rectangular dissection. Thus any valid orientation of S is the oriented region-maximal line incidence map of a trivalent rectangular dissection obtained from D by a sequence of operations of the type shown in figure 4.22 which reverses the sense of the pinwheel and rotates the dissection D' through an angle π , to form $\pi D'$, or by rotating the whole of D through π . This rotation of the whole of D corresponds to reversing the directions on all edges in S . The following proposition has thus been demonstrated.

Proposition 4.2. A validly oriented s-map with all z-vertices of degree four is the oriented region-maximal line incidence map of a trivalent rectangular dissection.

The oriented region-maximal line incidence maps of trivalent rectangular dissections have been considered previously by Flemming (1978) in the equivalent form of wall representations. In Flemming (1977) the number of wall representations with $n \leq 10$ regions is given by establishing a recurrence based on the fact that each trivalent rectangular dissection has its 'top left corner' rectangle of one of the types shown in figure 4.23. Thus the number of validly oriented s-maps with $n \leq 11$ z-vertices all degree four is given (table 4.1). If the wall representation is considered as an ornamentation of the rooted maximal line adjacency map then the effect of the ornamentation can be gauged by comparing the number of validly oriented s-maps with n z-vertices and the number of rooted quadrangulations of a quadrilateral with n internal quadrangular faces (table 4.2). The effect of the ornamentation

is seen to be small for $n \leq 9$ because there are few reversible submaps.

The generalization of proposition 4.2 for validly oriented s-maps is now stated and proved using the result of proposition 4.2.

Proposition 4.3. A validly oriented s-map is the oriented region-maximal line incidence map of a connected trivalent rectangular shape.

Proof. Consider the validly oriented s-maps with $|Z| = m$, suppose that the proposition holds for all $m + 2 < |X| + |Y| \leq n$. Let $S(n+1)$ be a validly oriented s-map with $|X| + |Y| = n + 1$. At least one of the configurations in figure 4.24 (up to interchange of x and y labels) occurs in $S(n+1)$. If one of the operations in figure 4.25 is applied to $S(n+1)$, it yields an oriented map with $|Z| = m$ and $|X| + |Y| = n$. It is validly oriented. By applying equation (14) or (15) no two faces share both an x and a y-vertex and the map is an s-map. Thus by induction it is the oriented region-maximal line incidence map of a connected trivalent rectangular shape of type $\{b, c\}$. The operations in figure 4.26 (corresponding to those in figure 4.25) on this shape construct a connected trivalent rectangular shape whose region-maximal line incidence map is $S(n+1)$. To complete the induction each validly oriented s-map with $|X| + |Y| = m + 2$ is the oriented region-maximal line incidence map of a trivalent rectangular dissection (proposition 4.2).

Consider a validly oriented s-map and specify positive numbers on the directed edges such that the sums on indirected and

outdirected (x,z) edges (and (y,z) edges) are equal at each vertex. These ornamented s-maps specify rectangular shapes completely, where the numbers correspond to dimensions, in the sense given below.

Consider a validly oriented s-map S , with vertices labelled $1, 2, \dots, n$, (x,z) edges labelled $1, 2, \dots, n-2$ and (y,z) edges labelled $1, 2, \dots, n-2$. Let $U = \{u_{ij}\}$ and $V = \{v_{ij}\}$, $1 \leq i \leq n$, $1 \leq j \leq n-2$, be matrices defined by

$$\begin{aligned} u_{ij} &= +1 \text{ } (-1) \text{ if the } (x,z) \text{ edge } j \text{ is indirected (outdirected)} \\ &\text{ at the vertex } i, \\ &= 0 \text{ otherwise,} \end{aligned}$$

with v_{ij} defined similarly. Suppose there are positive numbers \hat{x}_j and \hat{y}_j on the (x,z) edge j and the (y,z) edge j , $1 \leq j \leq n-1$. The conditions for zero sums at each vertex are expressed by the equations

$$\hat{\underline{x}} U^* = 0 \quad \hat{\underline{y}} V^* = 0 \quad (16)$$

where $\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n-2})$ and $\hat{\underline{y}} = (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{n-2})$, and U^* , V^* are the transpose matrices of U and V , respectively.

In order to see how the numbers $\hat{\underline{x}}$ and $\hat{\underline{y}}$ determine dimensioned rectangular shapes further numbers are derived on the x and y -vertices. Let the faces of S be labelled $1, 2, \dots, n-2$ and be given a counterclockwise orientation. Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ $1 \leq i \leq n-2$, $1 \leq j \leq n-2$ be matrices where

$$\begin{aligned} a_{ij} &= +1(-1) \text{ if the } (x,z) \text{ edge } j \text{ is positively (negatively)} \\ &\text{ in the face } i, \\ &= 0 \text{ otherwise,} \end{aligned}$$

and with b_{ij} defined similarly. Let the x and y -vertices be

labelled $1, 2, \dots, p$ and $1, 2, \dots, q$ respectively. Let $C = \{c_{ij}\}$, $1 \leq i \leq n - 2$, $1 \leq j \leq p$, and $D = \{d_{ij}\}$, $1 \leq i \leq n - 2$, $1 \leq j \leq q$ be matrices defined by

$$\begin{aligned} c_{ij} &= +1 \text{ if the } x\text{-vertex } j \text{ is in face } i \\ &= 0 \text{ otherwise} \end{aligned}$$

and the d_{ij} defined similarly. Numbers $\underline{x} = (x_1, x_2, \dots, x_p)$ and $\underline{y} = (y_1, y_2, \dots, y_q)$, up to an arbitrary constant, can be assigned to the x and y -vertices according to the equations

$$\underline{y} D^* A = \underline{x} \quad \underline{x} C^* B = \underline{y}. \quad (17)$$

A rectangular shape can be constructed from the numbers \underline{x} and \underline{y} on the x and y -vertices of S . Consider the x -vertex k together with the y -vertices i and j as shown in figure 4.27. This represents a line segment with endpoints (y_i, x_k) and (y_j, x_k) . The y -vertices are considered similarly. The line segment corresponding to the x -vertex k has its endpoints on the line segments corresponding to y -vertices i and j . The shape composed of these line segments is a connected rectangular shape of type $\{b, c\}$. However, the line segments are not necessarily maximal lines. The line segment corresponding to the x -vertex k is not a maximal line if and only if the configuration in figure 4.28 occurs where $x_k = x_m$. Similarly for the y -vertices. If each x and y -vertex corresponds to a maximal line then the corresponding rectangular shape is trivalent and S is its oriented region-maximal line incidence map.

Consider the sets $\{x_1, x_2, \dots, x_p\}$ and $\{y_1, y_2, \dots, y_q\}$ consisting of the components of \underline{x} and \underline{y} . If $x_i = x_j$ then the line segments corresponding to the x -vertices i and j are aligned. The alignment of two line segments which form part of a single

maximal line is a particular case of an alignment. Note that if the x -vertices i and j are joined by an (x,z) directed path from i to j (j to i) then $x_i > x_j$ ($x_j > x_i$). If the sets $\{x_1, x_2, \dots, x_p\}$ and $\{y_1, y_2, \dots, y_q\}$ consist of m and ℓ distinct numbers respectively, then the corresponding shape is a connected minimal (ℓ, m) rectangular shape. If $\ell = q$ and $m = p$ then it is a trivalent, non-aligned, minimal (ℓ, m) rectangular shape.

Consider a validly oriented s -map with all z -vertices degree four. It is the oriented region-maximal line incidence map of a trivalent rectangular dissection. Consider the oriented map obtained by removing y -vertices and contracting (x,z) directed edges. The result is called the oriented x -network of the dissection. The oriented y -network is defined similarly. The oriented x and y -networks are used by Brooks et al (1940) to examine the dissection of rectangles into squares. The underlying maps are the x and y -network representations, respectively with the directed edges corresponding to the distinguished z -vertex as root edge. Note that the oriented x and y -networks of a trivalent rectangular dissection each define the validly oriented s -map.

Consider a 2-vertex connected rooted plane map M with edges labelled $1, 2, \dots, p+1$, where $p+1$ is the root edge. Consider an assignment of directions to the non-root edges together with pairs of non-zero numbers (a_i, b_i) $1 \leq i \leq p+1$ such that the sum of the a_i on indirected and outdirected edges at each vertex are equal, and the sum of the b_i , taken (according to a given direction of traversal) around a face is zero. Also let the directed root edge (v,w) be the only outdirected edge at v and the only indirected edge at w . These are the Kirchhoff laws and the result is called

a Kirchhoff chain on M . If the directions are chosen such that the numbers $\{a_1, a_2, \dots, a_p, a_{p+1}\}$ and $\{b_1, b_2, \dots, b_p, -b_{p+1}\}$ are positive, then a K-orientation on M is defined.

In a K-orientation each directed circuit contains the root edge. Also if the vertices are labelled according to figure 4.29 there are precisely two "1" labels at each vertex and precisely two "0" labels around the interior of each face. Consider the rooted dual of M with directions and number pairs assigned to the edges as shown in figure 4.30. They form the dual Kirchhoff chain on M^* . The derived map of M with $(W(F), W(E))$ edges removed and directions assigned to the edges according to the K-orientations on M and M^* , together with $W(V), W(F)$ and $W(E)$ vertices labelled x, y and z , respectively, is a validly oriented s-map. The K-orientations on M and M^* are the oriented x and y -networks. If the numbers $(a_1, a_2, \dots, a_{p+1})$ and $(b_1, b_2, \dots, -b_{p+1})$ are assigned to the (x, z) and (y, z) edges respectively in this validly oriented s-map then equation (16) is satisfied and a rectangular dissection is specified.

To summarize, each K-orientation of a 2-vertex connected rooted plane map is the oriented x or y -network of a trivalent rectangular dissection, and conversely. Each Kirchhoff chain specifies a rectangular dissection completely.

For a given 2-vertex connected rooted plane map the freedom to give K-orientations corresponds to the freedom to give valid orientations to the s-maps which have all z -vertices degree four. Thus given a K-orientation every other K-orientation can be derived

by reversing directions on the internal edges of configurations of the type shown in figure 4.31. They are the embedded Wheatstone Bridges (Duffin 1965) and correspond to the pinwheel configurations in the corresponding rectangular dissections (see figure 4.21).

4.4.4 Region adjacency

Consider a validly oriented s-map S . Construct a plane map T from S in the following way. Insert diagonals in the faces according to the scheme in figure 4.32. The diagonals are labelled x or y . Remove the x and y -vertices. The faces in the resulting map which correspond to x -vertices (y -vertices) have exactly two x -edges (y -edges) in their boundaries. Triangulate the faces corresponding to x -vertices (y -vertices), which have degree greater than three, using x -edges (y -edges) such that no triangular face has all x -edges or all y -edges in its boundary (figure 4.33). Also replace the digon faces which have both boundary edges with the same label by a single edge with that label. Finally, expand z^* according to the scheme in figure 4.34 and let the new face be the distinguished face of T . Plane maps constructed in this way from validly oriented s-maps are called t -maps.

T is the region adjacency map of a connected trivalent rectangular shape (figure 4.35) whose validly oriented region-maximal line incidence map is S . The external region is considered to be divided into subregions at the corners of the shape with $3\pi/2$ external angle. The x and y -edges represent adjacency at "horizontal" and "vertical" line segments. The different t -maps, derivable, by the above construction, from a given validly oriented s-map, S , correspond to the freedom to order the maximal lines adjacent to a given maximal line in the connected trivalent

rectangular shapes whose oriented region-maximal line incidence maps are S .

Finally consider the region adjacency maps of the generalized trivalent rectangular dissections. These are the t -maps derived from the validly oriented s -maps with all non-distinguished vertices degree four. These t -maps are validly coloured simple $[r,s]$, $r \geq 1$, $s \geq 4$, triangulations (see section 3.4). The converse holds by proposition 3.2.

Proposition 4.4. Each validly coloured simple $[r,s]$, $r \geq 1$, $s \geq 4$, triangulation is the region adjacency map of a generalized trivalent rectangular dissection.

4.4.5 Conclusion

The representations developed in this section have been used in various approaches to floor plan design based on rectangular shapes. Attention has been largely restricted to trivalent rectangular dissections. The region adjacency representation forms the basis of Grason's thesis (1970) on the automatic generation of floor plans. A partial region adjacency map is completed and dimension constraints satisfied. The region adjacency representation is also used extensively by Mitchell, Steadman and Liggett (1976) in their work on small rectangular plans. The network representation is used by Teague (1968) who generalizes it to represent configurations of rectangular parallelepipeds. Finally, the oriented region-maximal line representation is used by Flemming (1978) where it is called the wall representation.

The representations considered in section 4.4 show how to

consider rectangular shapes at various level of detail. Each representation considers different elements and relations between them. The representations are considered as ornamentations of the maximal line adjacency maps. This demonstrates the connections between the representations. In the ornamentation of numbering the (x,z) and (y,z) edges in the oriented region-maximal line incidence maps the full description of the rectangular shapes is recovered. In a sense the inverse of the original representation has been constructed.

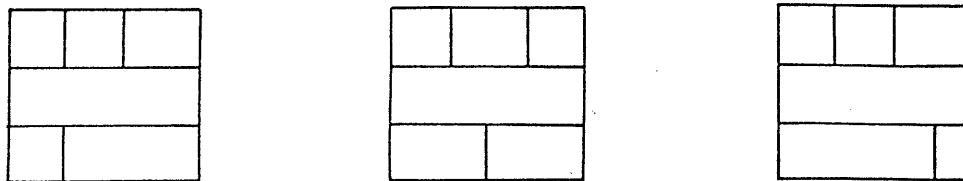


FIGURE 4.1 A $[6,4]$ rectangulation with different alignments.

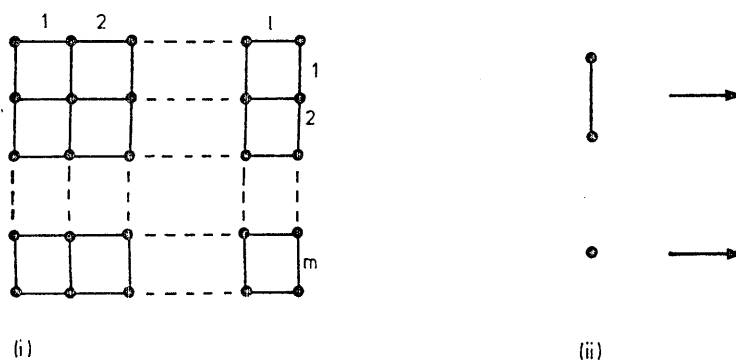


FIGURE 4.2 Construction of (\underline{l}, m) rectangular shapes.

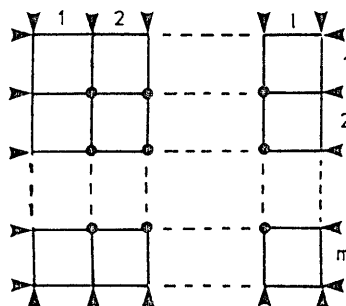


FIGURE 4.3 A marked (\underline{l}, m) rectangular grating.

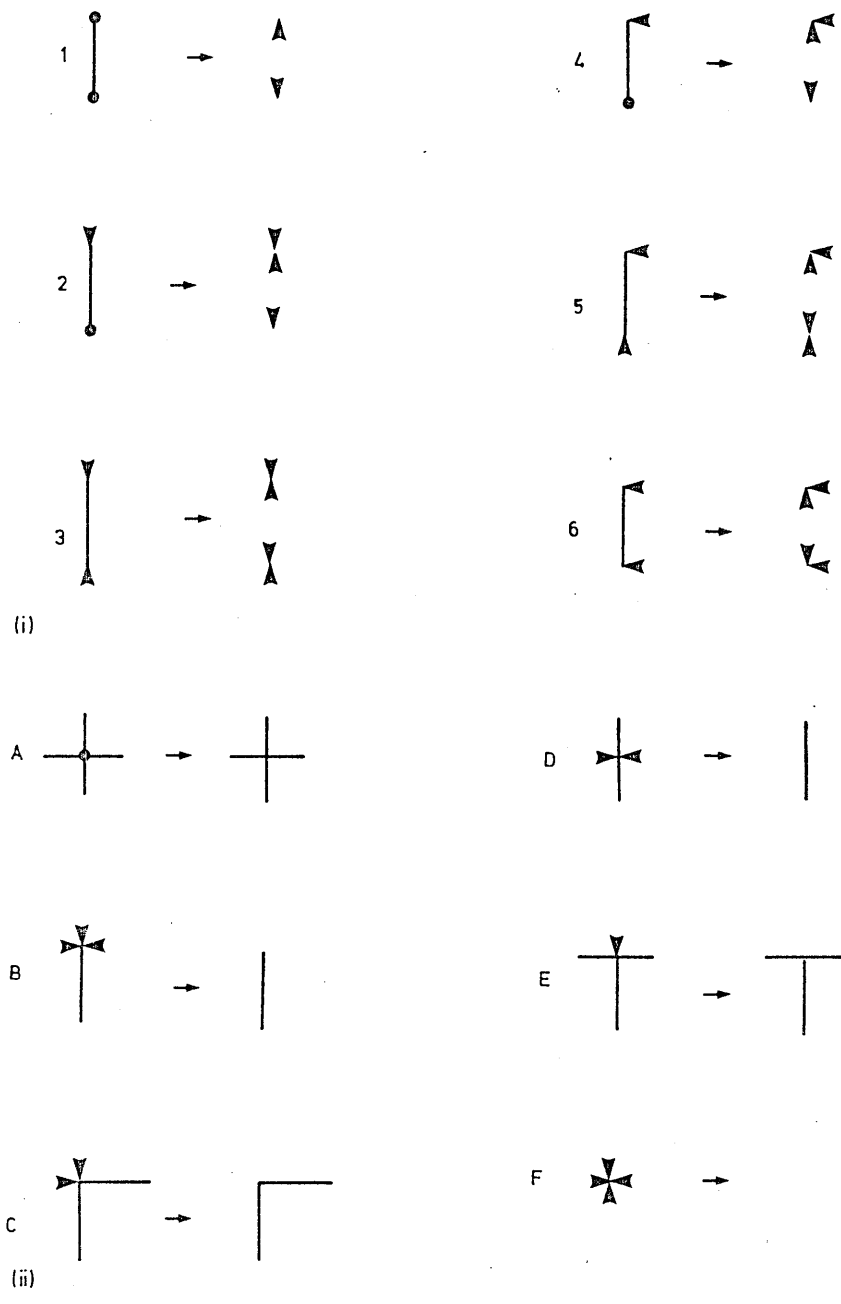


FIGURE 4.4 The rules to construct rectangular shapes.

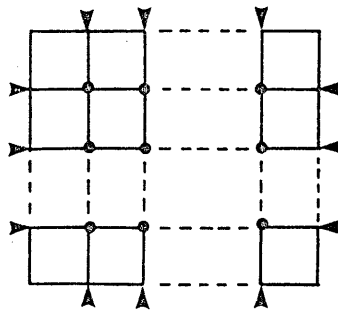


FIGURE 4.5 A marked (\underline{l}, m) rectangular grating.

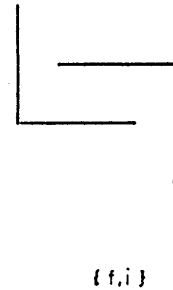
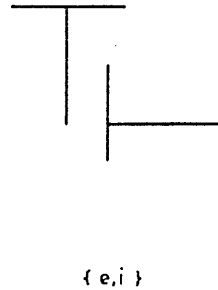
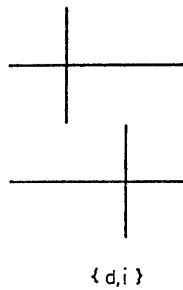


FIGURE 4.6 Rectangular shapes of types $\{d,i\}$, $\{e,i\}$ and $\{f,i\}$.

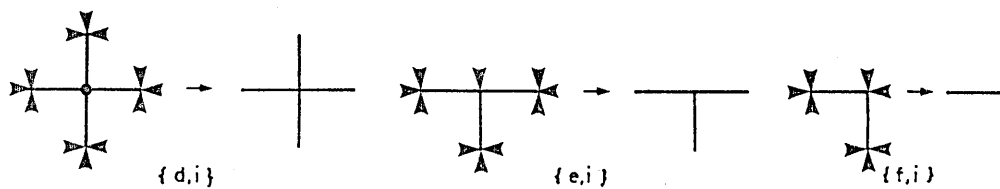


FIGURE 4.7 Erasing rules for rectangular shapes of types $\{d,i\}$, $\{e,i\}$ and $\{f,i\}$.

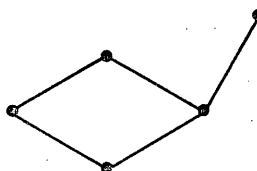
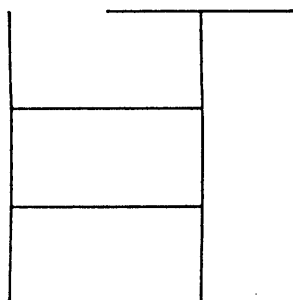
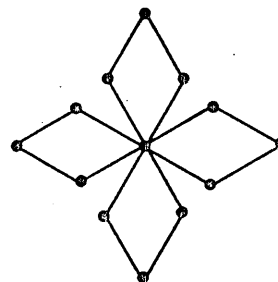
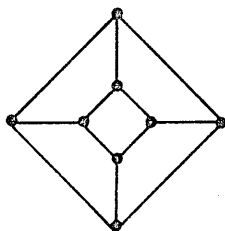
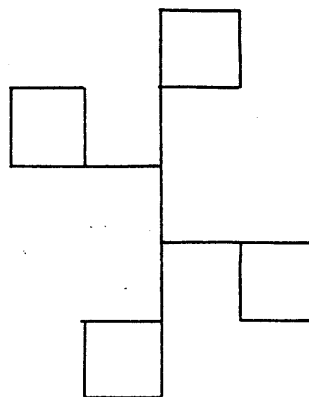
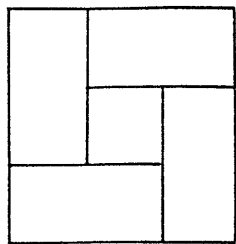


FIGURE 4.8 Maximal line adjacency maps.

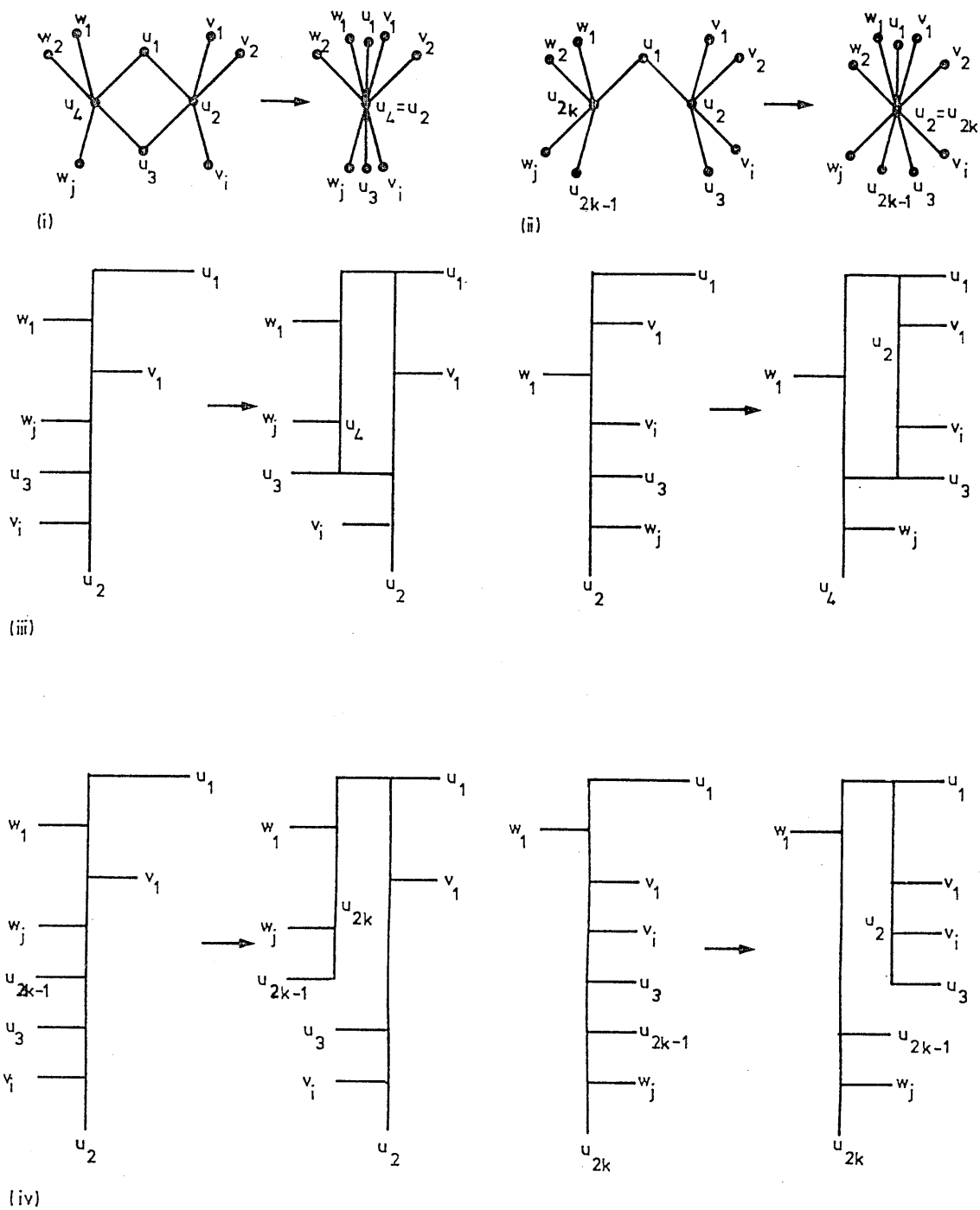


FIGURE 4.9 Construction of a connected trivalent rectangular shape whose maximal line adjacency maps is $Q(n+1)$.

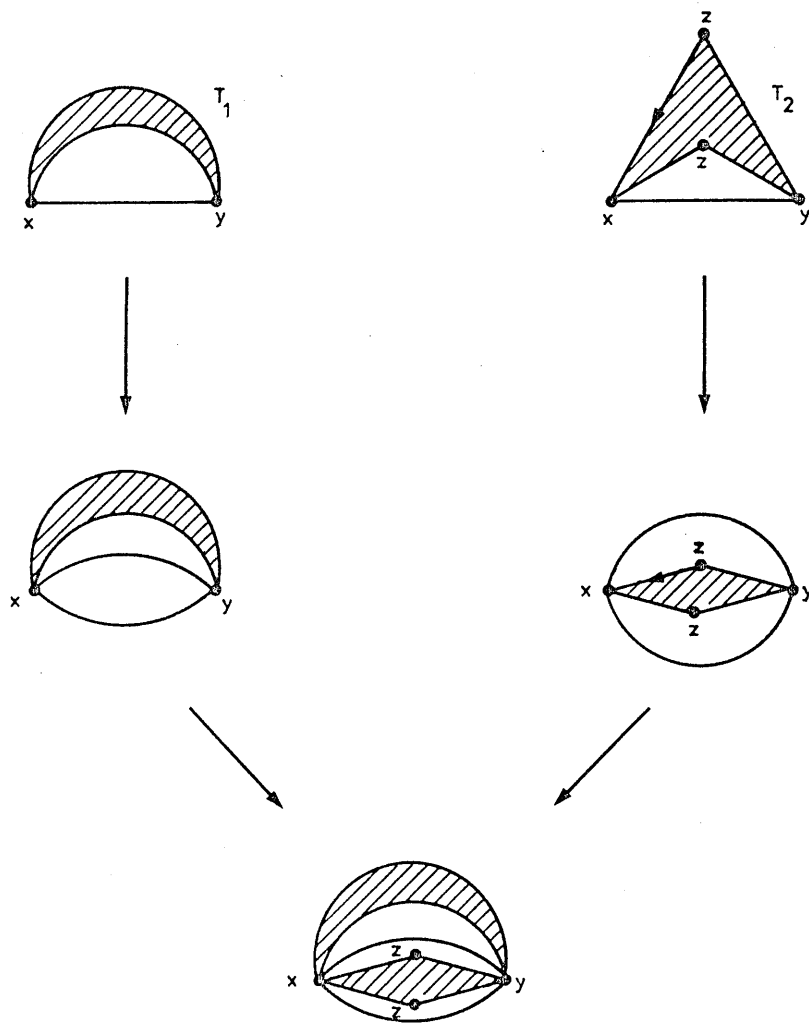
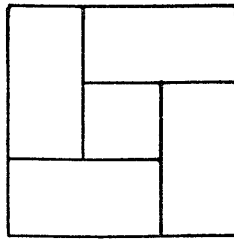
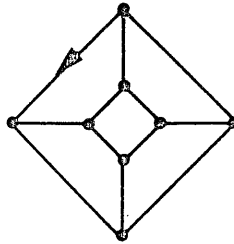


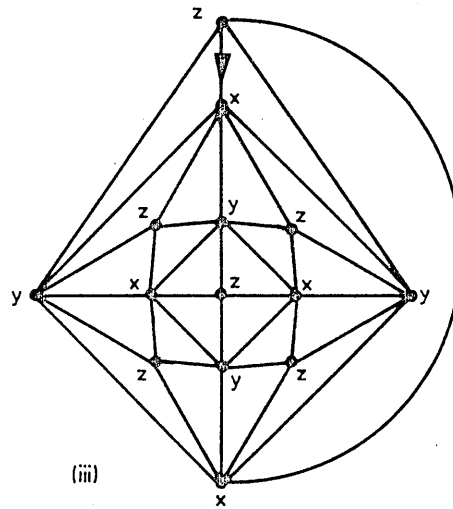
FIGURE 4.10 Splitting an (x,y) -edge in a rooted z -map, T_1 , and inserting a 3-coloured rooted triangular map, T_2 .



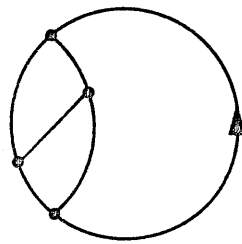
(i)



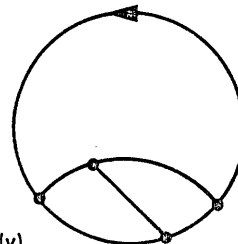
(ii)



(iii)



(iv)



(v)

FIGURE 4.11 (i) A trivalent rectangular dissection, (ii) the corresponding rooted q-map, (iii) the rooted z-map, (iv) the x-network and (v) the y-network.

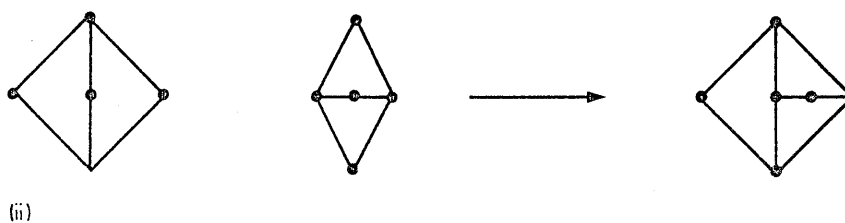
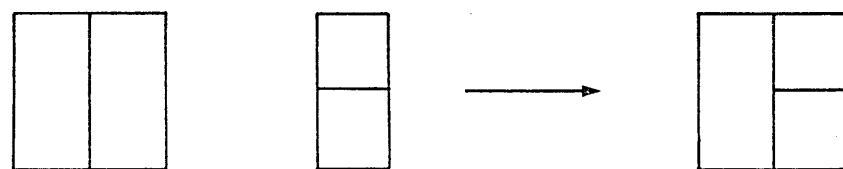
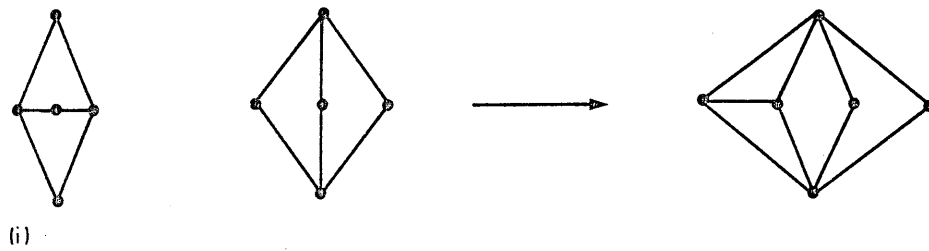
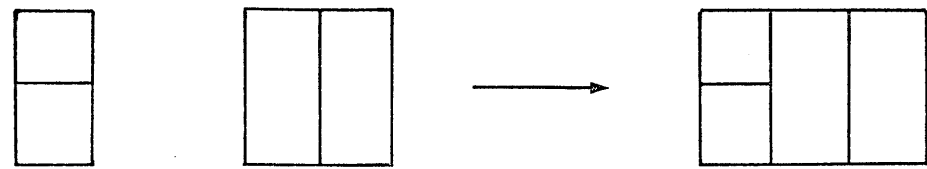


FIGURE 4.12 Addition of rectangular dissections in terms of the maximal line adjacency maps: (i) quadrangulation of the external face and (ii) quadrangulation of an internal face.

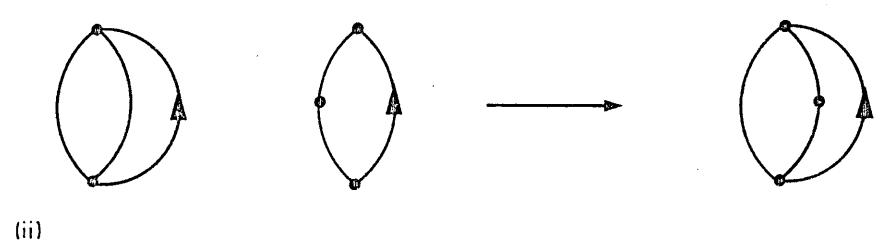
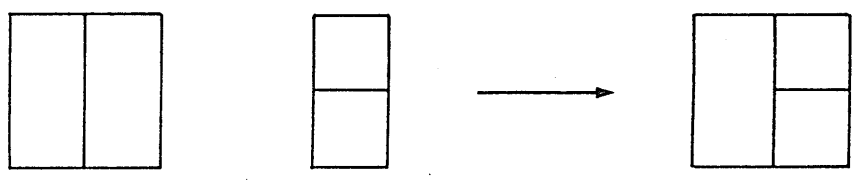
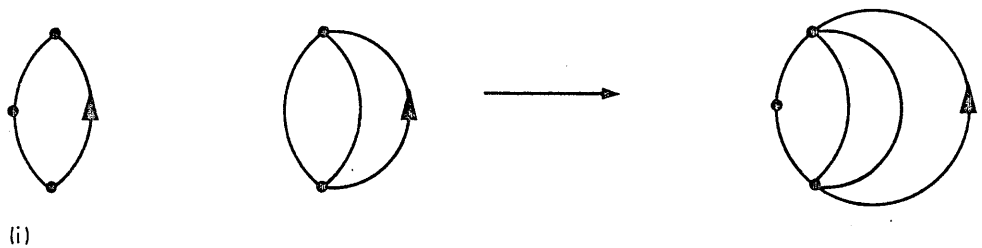
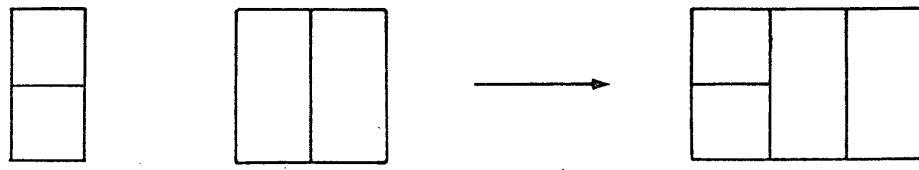
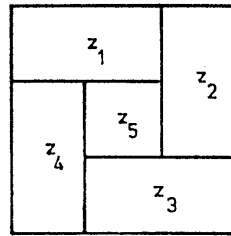
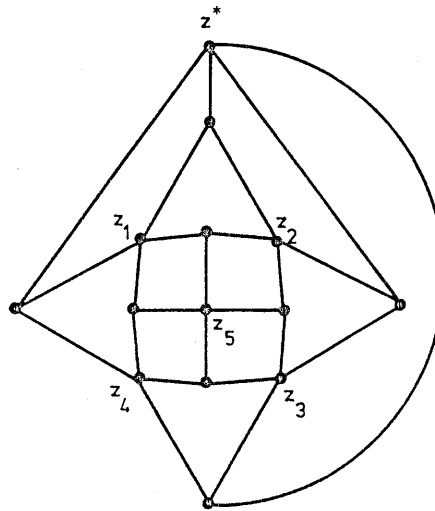


FIGURE 4.13 Addition of rectangular dissections in terms of the x-networks: an operation (i) at the root edge and (ii) at a non-root edge.

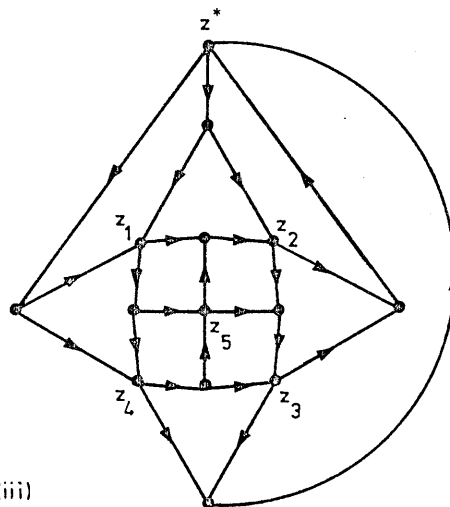


z^*

(i)



(ii)



(iii)

FIGURE 4.14 (i) A trivalent rectangular dissection, (ii) the region-maximal line incidence map and (iii) the oriented region-maximal line incidence map.

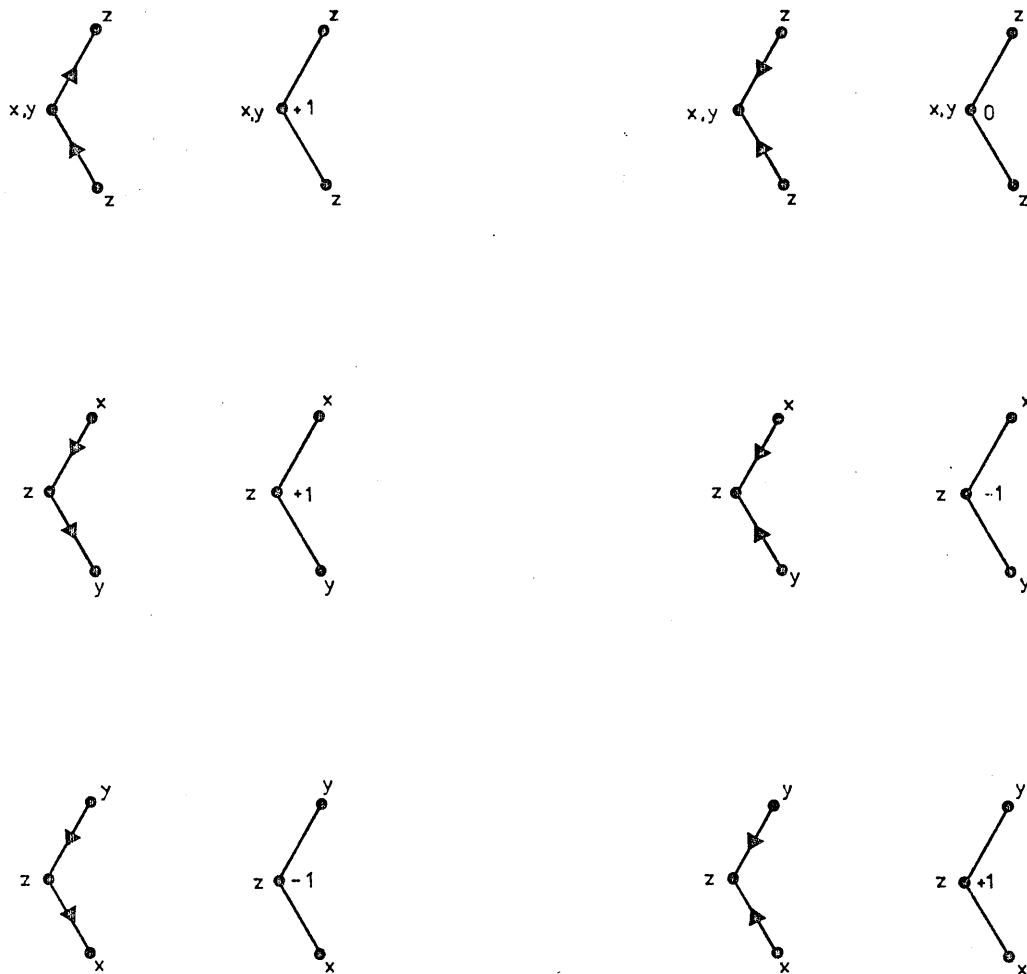


FIGURE 4.15 Labels in the faces of an oriented s-map.

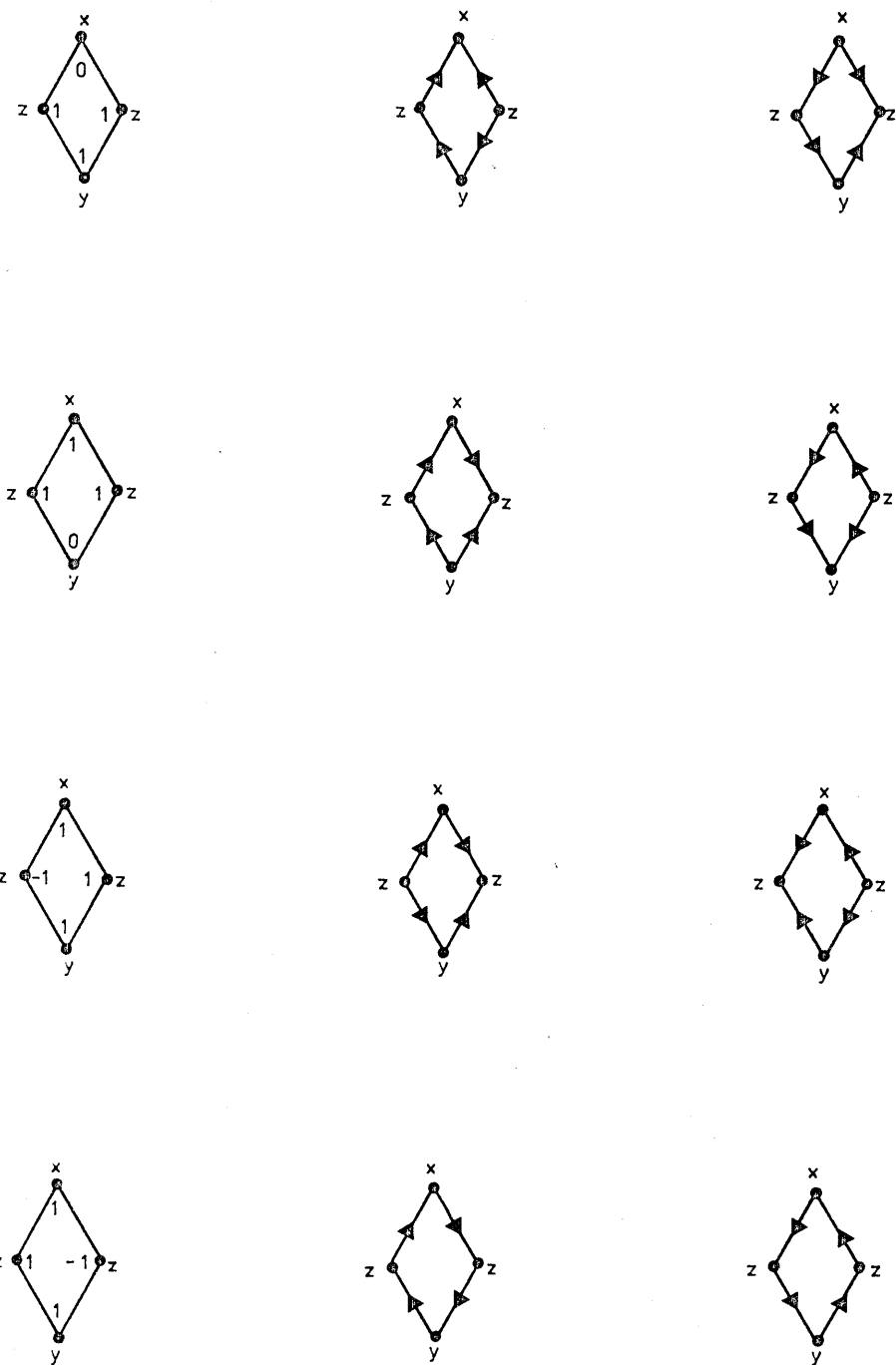


FIGURE 4.16 Possible faces in a validly oriented s-map and the associated labellings.

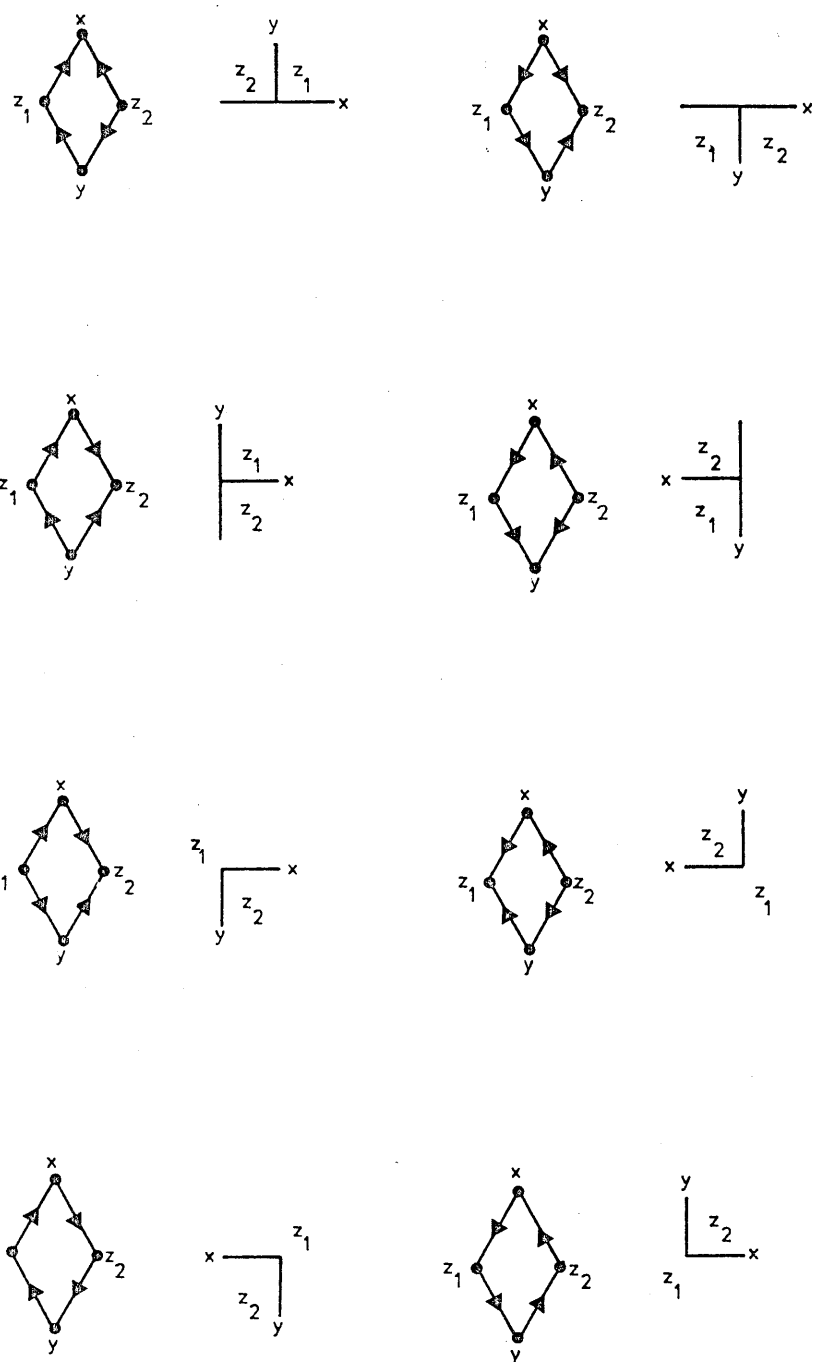


FIGURE 4.17 Faces in an oriented region-maximal line incidence map and corresponding intersections of maximal lines.

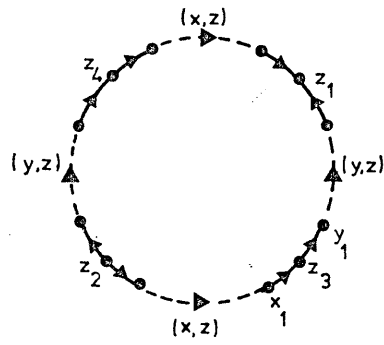


FIGURE 4.18 Configuration in S .

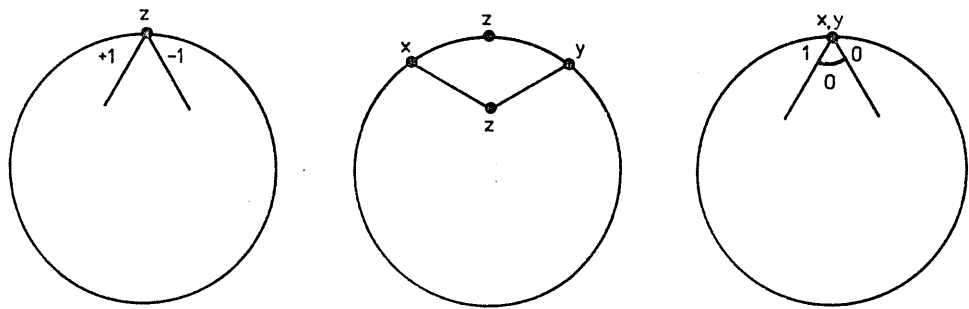


FIGURE 4.19 Interior labels at vertices on the boundary of a reversible submap.

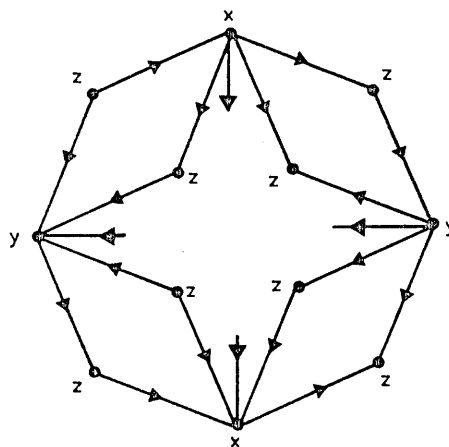


FIGURE 4.20 Reversible submap in a validly oriented s -map with all z -vertices of degree four.

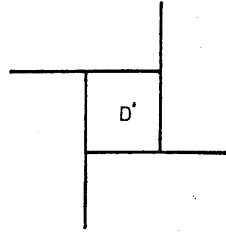


FIGURE 4.21 Pinwheel configuration.

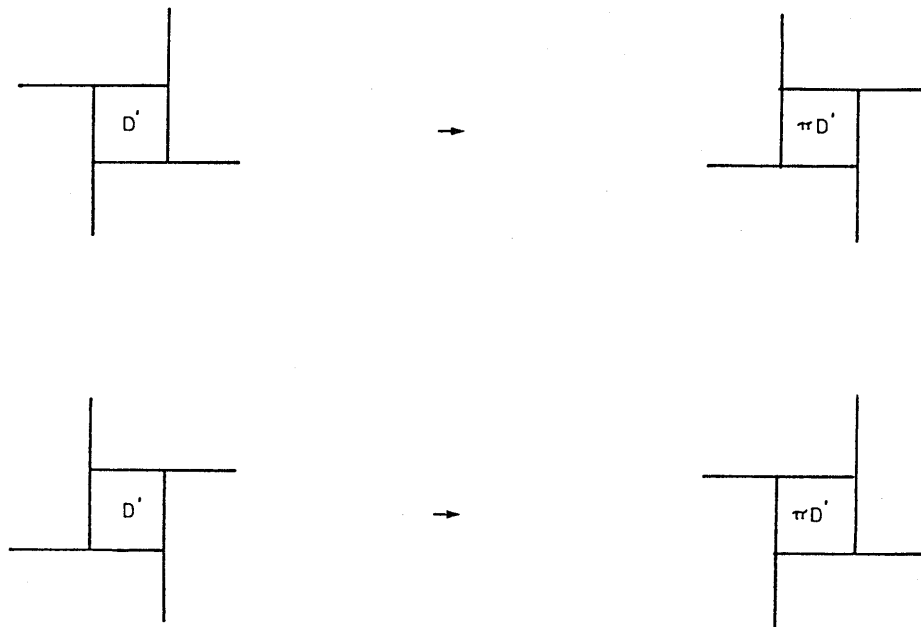


FIGURE 4.22 Operations on a rectangular dissection which correspond to reversing the edge directions in a reversible submap.

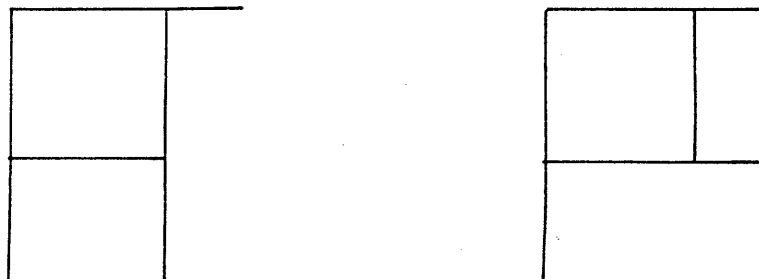


FIGURE 4.23 "Top left corner" rectangle in a rectangular dissection.

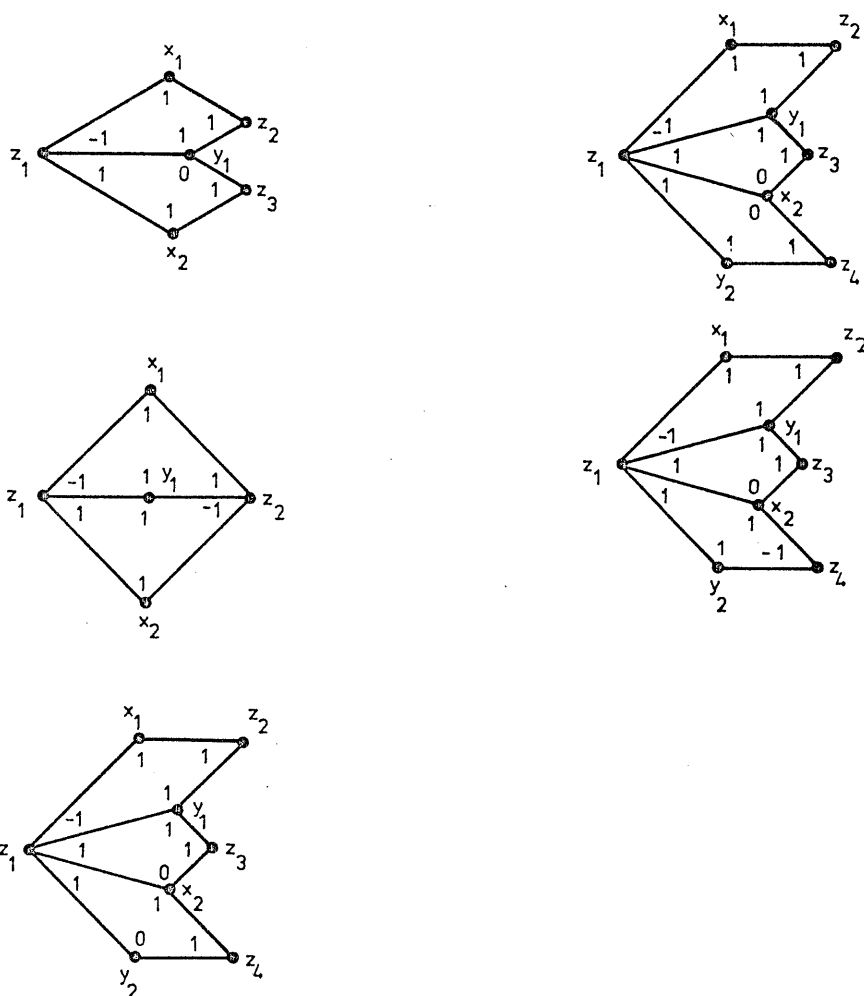


FIGURE 4.24 Configurations in $S(n+1)$.

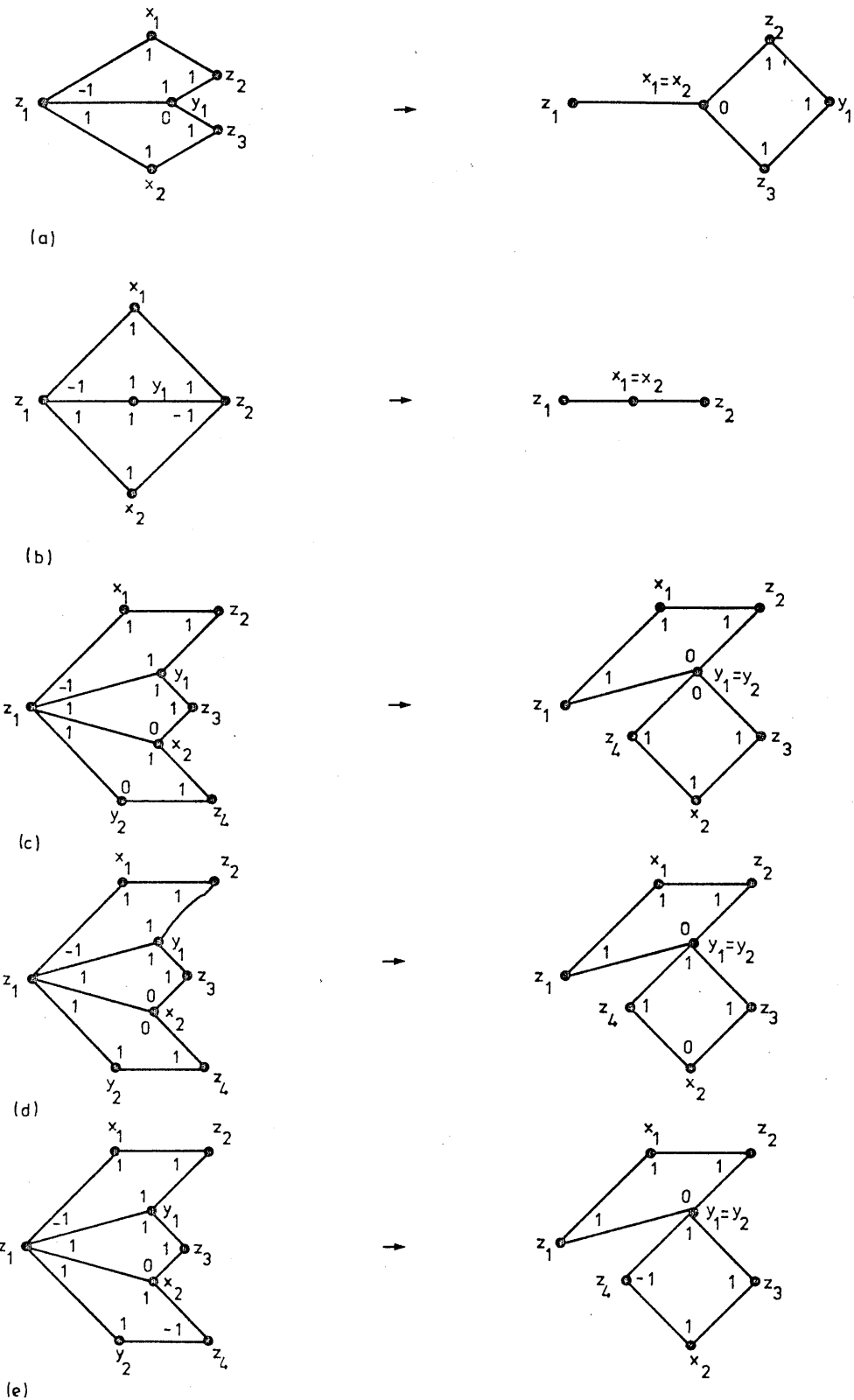


FIGURE 4.25 Identification of vertices in $S(n+1)$.

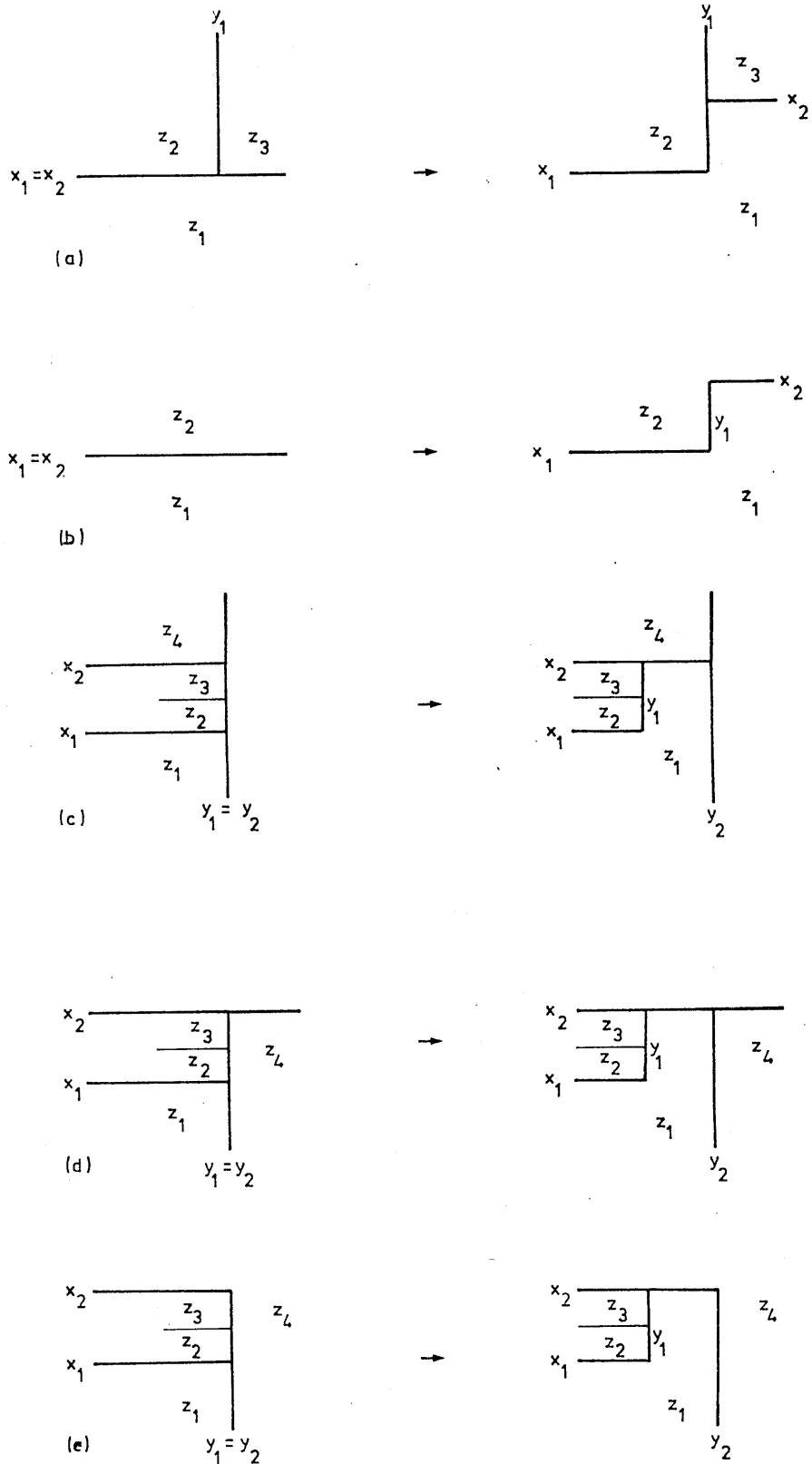


FIGURE 4.26 Construction of a connected trivalent rectangular shape whose oriented region-maximal line incidence map is $S(n+1)$.

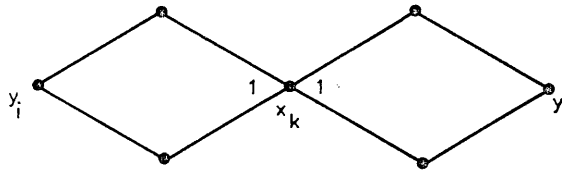


FIGURE 4.27 The x-vertex k , whose corresponding line segment has endpoints (y_i, x_k) and (y_j, x_k) .

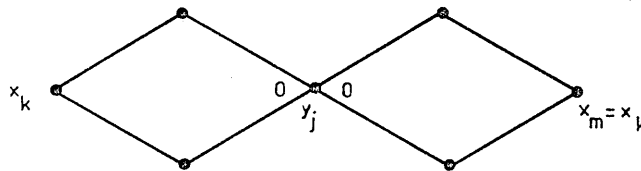


FIGURE 4.28 The x-vertices k and m whose corresponding line segments are part of a single maximal line.

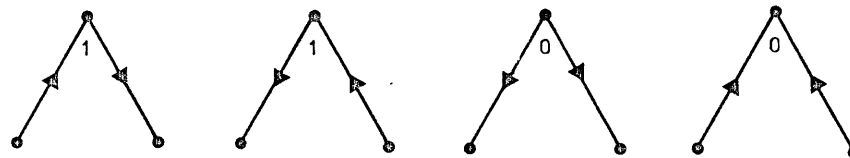


FIGURE 4.29 Labels at the vertices in a k -orientation.

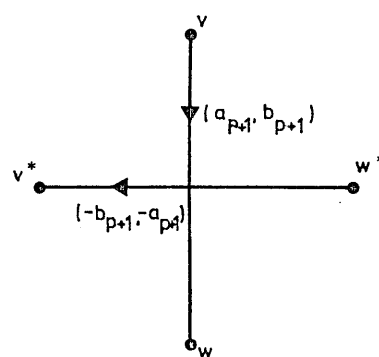
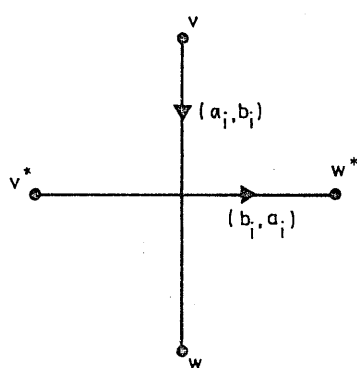


FIGURE 4.30 Dual Kirchhoff chain.

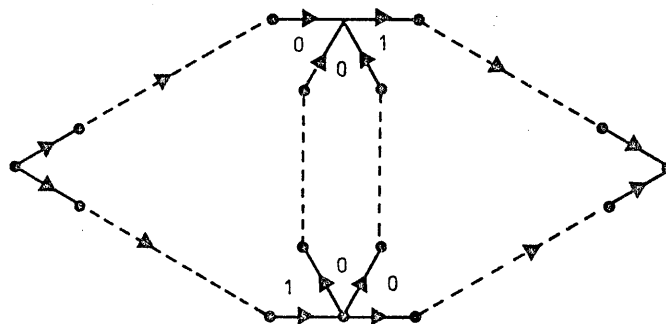
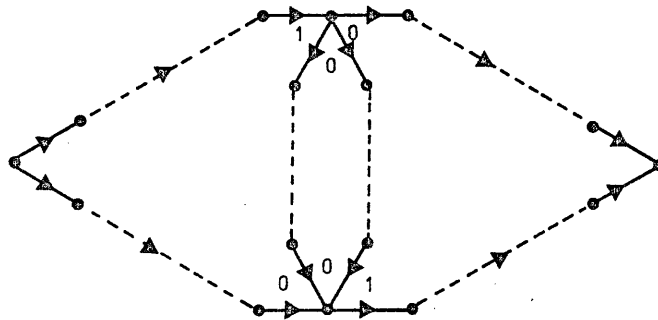


FIGURE 4.31 Embedded Wheatstone bridges.

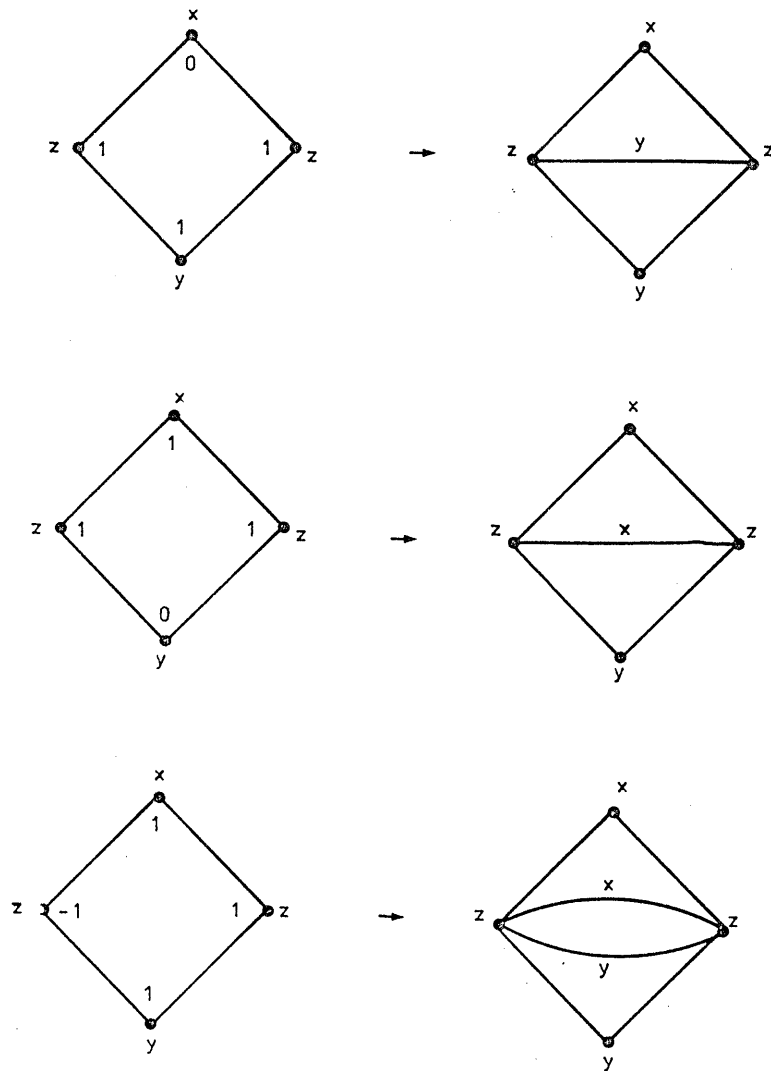


FIGURE 4.32 Diagonals in faces of S .

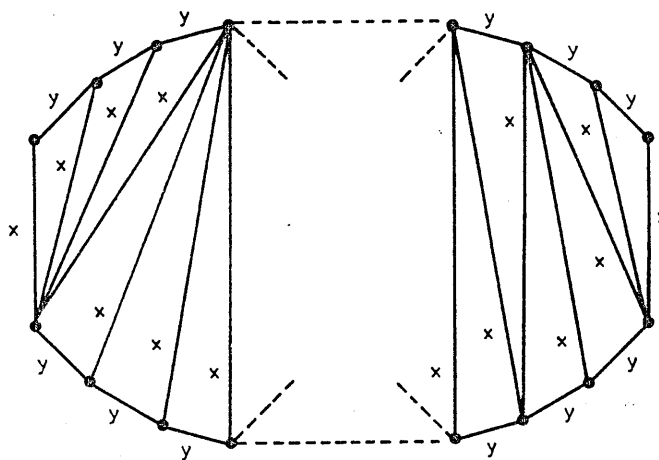


FIGURE 4.33 Triangulation of faces formed by diagonals.

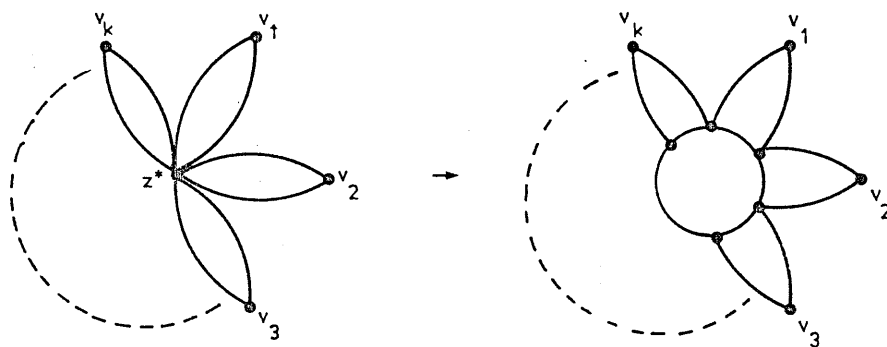


FIGURE 4.34 Expansion of z^* to form distinguished face.

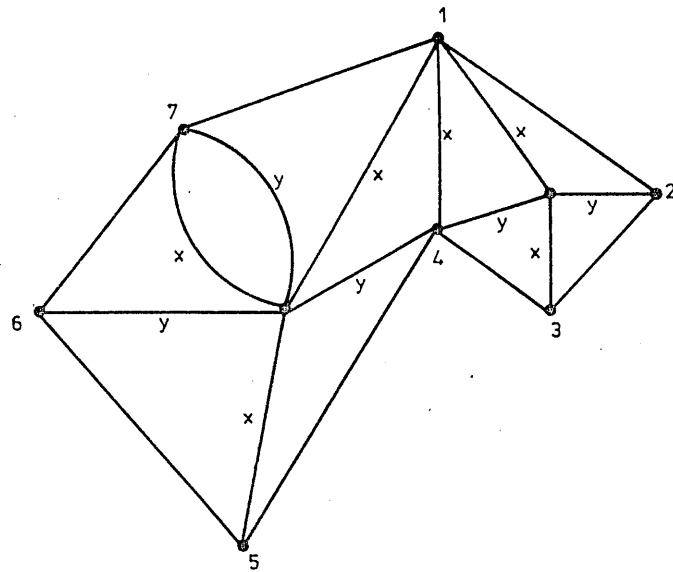
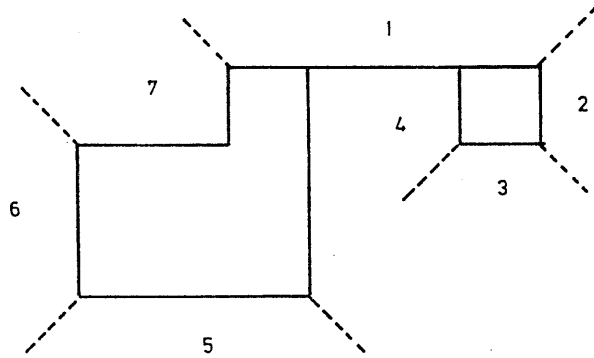


FIGURE 4.35 Region adjacency map.

n	2	3	4	5	6	7	8	9	10	11
	1	2	6	22	92	422	2074	10 754	58 202	326 240

TABLE 4.1 Validly oriented s-maps with $n \leq 11$ z-vertices, all degree four.

n	2	3	4	5	6	7	8	9	10	12
	1	2	6	22	91	408	1938	9614	49 335	260 130

TABLE 4.2 Rooted quadrangulations of a quadrilateral with $n \leq 11$ quadrangular faces.

5.1 Introduction

A kinematic chain is a collection of rigid bodies whose mutual connections, if any, are the constraints on their motion. Representations are developed for kinematic chains in terms of the incidences (or connections) between the links (or rigid bodies) composing the chain. Each representation corresponds to many geometric realizations, obtained by specifying the metrical properties of the links and their connections. This investigation is mostly concerned with the incidence patterns among the links.

The incidence patterns broadly correspond to the "kinematic structure" described in Buchsbaum and Freudenstein (1970) who argue there that "The separation of kinematic structure from functional considerations can be useful in the conceptual stages of mechanical design, in surveying potentially useful classes of mechanisms, and in clarifying structural similarities between functionally different mechanical embodiments."

Reuleaux (1876) distinguished six lower kinematic pairs, namely the screw, revolute, prismatic, cylindrical, spherical and planar pairs, which all possess surface contact between their two component links. These are some of the possible types of incidence between links in a kinematic chain. Attention will be largely restricted to these lower pairs although consideration will be given to pairs which are effectively constructed from collections of these lower pairs forming kinematic pairs in their own right. For example two links may be connected via two revolutes and an

intermediary link (figure 5.1), which can be considered as a kinematic pairing of the two links.

A coplanar kinematic chain (KC) has revolute and prismatic pairs with the axes of the revolute pairs parallel and the axes of the prismatic pairs in planes perpendicular to the axes of the revolute pairs. Any other type of kinematic chain will be called a spatial KC. In this chapter coplanar KCs are considered. Attention is restricted to coplanar KCs with only revolute pairs for these provide a relatively straightforward example to illustrate the representation and construction of kinematic chains in terms of the incidence structure of the component links.

The fundamental incidence structure of a coplanar kinematic chain is given by the revolute pairs. However, the revolute pairs are grouped by virtue of the fact that some of them may continuously share a common axis throughout the motion of the KC. This grouping of revolute pairs is essentially a geometric property of the KC and provides a further incidence structure. Suppose the groups of revolute pairs are called joints then links can be considered incident at joints. For example in a four cylinder internal combustion engine the crankshaft and connecting rods are often incident at four revolute pairs grouped as two joints (figure 5.2). The links ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are incident to the link ℓ_5 at the revolute pairs r_1, r_2, r_3 and r_4 , respectively. The revolute pairs $\{r_1, r_4\}$ and $\{r_2, r_3\}$ form two joints j_1 and j_2 , say. The links $\{\ell_1, \ell_4, \ell_5\}$ are incident in pairs at j_1 and the links $\{\ell_2, \ell_3, \ell_5\}$ are incident in pairs at j_2 .

A coplanar KC is described by the links, revolute pairs and

the groups of revolute pairs which form joints. The incidence structure between links at joints is derived. However, a joint may also be specified by links which are incident at the joint. Thus an equivalent description of a coplanar KC can be given which gives the two incidence structures, of links at revolute pairs and of links at joints, explicitly. These two descriptions are investigated in the next section.

5.2 Coplanar KCs with only revolute pairs

A set of revolute pairs continuously sharing a common axis, which is maximal in the sense that no other revolute pairs share this axis, forms a joint. A joint may also be represented by the links in the revolute pairs forming the joint. A joint represents a geometric condition on a set of revolute pairs, namely that they continuously share a common axis. The incidence patterns of links determined by revolute pairs are augmented by the grouping of revolute pairs to form joints. It is these extended incidence patterns which will be investigated.

The extended incidence patterns may be described in essentially two ways. In the first description (the (L,R,J) description) the joints are considered as groups of revolute pairs. In the second description (the (L,J,R) description) joints are considered as collections of links and revolute pairs are pairings of links in the joints. The conditions on the collection of revolute pairs forming a joint in the (L,R,J) description are similar to the conditions on the pairing of links in a joint in the (L,J,R) description. The (L,R,J) description considers links and revolute pairs as primary elements and the specific grouping of revolute

pairs into joints as an ornamentation. The (L,J,R) descriptions considers links and joints as the primary elements and the specific pairing of links in a joint into revolute pairs as an ornamentation.

5.2.1 The (L,R,J) description

A coplanar KC with only revolute pairs is an ordered triple (L,R,J) of links L , revolute pairs $R \subseteq \mathcal{P}_2(L)$ and joints $J \subseteq \mathcal{P}(R)$ such that if $L(j) = \{\ell \in L: \ell \in r \in j \text{ for some } r \in R\}$, $j \in J$ and $J(\ell) = \{j \in J: \ell \in L(j)\}$, $\ell \in L$ then

- 1) $\bigcup_{j \in J} j = R$, $j_1 \cap j_2 = \emptyset$ for all $j_1 \neq j_2 \in J$,
- 2) $|J(\ell_1) \cap J(\ell_2)| \leq 1$ for all $\ell_1 \neq \ell_2 \in L$,
- 3) the graphs $(L(j), j)$, are trees for all $j \in J$.

The ordered pair (L,R) is a graph with vertices L , and edges R , which represents the incidence between links at revolute pairs.

The revolute pairs which constitute a joint $j \in J$ induce a subgraph $(L(j), j)$ of (L,R) . In practice it is possible that the graphs $(L(j), j)$, $j \in J$, are not trees, although they must be connected.

A tree represents the minimum pairing between links at a joint.

In the example shown in figure 5.2, $L = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$, $R = \{(\ell_1, \ell_5), (\ell_2, \ell_5), (\ell_3, \ell_5), (\ell_4, \ell_5), (\ell_5, \ell_6)\} = \{r_1, r_2, r_3, r_4, r_5\}$, and $J = \{(r_1, r_4), (r_2, r_3), (r_5)\} = \{j_1, j_2, j_3\}$. The links incident to the joints are $L(j_1) = \{\ell_1, \ell_4, \ell_5\}$, $L(j_2) = \{\ell_2, \ell_3, \ell_5\}$ and $L(j_3) = \{\ell_5, \ell_6\}$. The joints at which the links are incident to other links are $J(\ell_1) = \{j_1\}$, $J(\ell_2) = \{j_2\}$, $J(\ell_3) = \{j_2\}$, $J(\ell_4) = \{j_1\}$, $J(\ell_5) = \{j_1, j_2, j_3\}$ and $J(\ell_6) = \{j_3\}$. The graphs $(L(j_1), j_1)$, $(L(j_2), j_2)$ and $(L(j_3), j_3)$ are subgraphs of (L,R) and are shown in figure 5.3.

There are two further points to note about the coplanar KCs for which the above description applies. First, they include collections of links whose graphs (L,R) are not connected, that is, collections of links composed of unconnected subcollections. Second, they exclude collections of links in which two links are incident at two distinct joints. Such pairs of links can have no relative motion.

The links incident at a joint $j \in J$ are $L(j)$ and the joints at which a link $\ell \in L$ is incident to other links are $J(\ell)$. Let $d(\ell) = |J(\ell)|$ be the degree of a link $\ell \in L$ and $d(j) = |L(j)|$ be the degree of a joint $j \in J$. The pair degree of a link $\ell \in L$ is $|\{r \in R : \ell \in r\}| = d'(\ell)$.

If (L,R,J) is a coplanar KC, then the triple (L,f,J) where $f : J \rightarrow \mathcal{P}(L)$ is a function defined by $f(j) = L(j)$, $j \in J$, represents the incidence of links at joints in (L,R,J) . There is a dual representation by the triple (J,f^*,L) where $f^* : L \rightarrow \mathcal{P}(J)$ is a function defined by $f^*(\ell) = J(\ell)$, $\ell \in L$. (L,f,J) is a system and (J,f^*,L) is its transpose (Graver and Watkins 1977). The system (L,f,J) is a set system. However, the transpose (J,f^*,L) is a set system if and only if $d(\ell) > 0$ for all $\ell \in L$.

A coplanar KC (L,R,J) can be represented by $(L,f(J),R)$ where $f(J) = \{f(j) : j \in J\}$. Also by condition (2) in the definition $j = \{r \in R : r = f(j) \cap r\}$, $j \in J$. This motivates the alternative description of a coplanar KC.

5.2.2 (L,J,R) description

A coplanar KC with only revolute pairs is a triple (L,J,R) of links L , joints $J \subseteq \mathcal{P}(L)$ and revolute pairs $R \subseteq \mathcal{P}_2(L)$ such that if $J(\ell) = \{j \in J : \ell \in j\}$, $\ell \in L$, and $R(j) = \{r \in R : r = r \cap j\}$, then

- (1) $\bigcup_{j \in J} R(j) = R$, $R(j_1) \cap R(j_2) = \emptyset$ for all $j_1 \neq j_2 \in J$,
- (2) $|J(\ell_1) \cap J(\ell_2)| \leq 1$ for all $\ell_1 \neq \ell_2 \in L$,
- (3) the graphs $(j, R(j))$ are trees for all $j \in J$.

Given a coplanar KC (L,R,J) then its (L,J,R) description is $(L, f(J), R)$ and given a coplanar KC (L,J,R) then its (L,R,J) description is $(L, R, R(J))$ where $R(J) = \{R(j) : j \in J\}$. $R(j)$ denotes the revolute pairs composed of links in joint $j \in J$.

For a coplanar KC (L,J,R) the ordered pair (L,R) is a graph which represents the incidence between links at revolute pairs. The links in a joint are incident at that joint. The joints at which a link $\ell \in L$ are incident to other links are $J(\ell)$. The ordered pair (L,J) is a set system which represents the incidence between links at joints. The function is the inclusion function. The transpose system is denoted by (J,L) . This is a set system if and only if $d(\ell) > 0$ for all $\ell \in L$.

In the example shown in figure 5.2 the (L,J,R) description has $L = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$, $J = \{(\ell_1, \ell_4, \ell_5), (\ell_2, \ell_3, \ell_5), (\ell_5, \ell_6)\} = \{j_1, j_2, j_3\}$ and $R = \{(\ell_1, \ell_5), (\ell_2, \ell_5), (\ell_3, \ell_5), (\ell_4, \ell_5), (\ell_5, \ell_6)\} = \{r_1, r_2, r_3, r_4, r_5\}$. The joints at which the links are incident to other links are $J(\ell_1) = \{j_1\}$, $J(\ell_2) = \{j_2\}$, $J(\ell_3) = \{j_2\}$, $J(\ell_4) = \{j_1\}$, $J(\ell_5) = \{j_1, j_2, j_3\}$ and $J(\ell_6) = \{j_3\}$. The revolute

pairs composed of links in the joints are $R(j_1) = \{r_1, r_4\}$, $R(j_2) = \{r_2, r_3\}$ and $R(j_3) = \{r_5\}$. The graphs $(j_1, R(j_1))$, $(j_2, R(j_2))$ and $(j_3, R(j_3))$ are subgraphs of (L, R) and are shown in figure 5.4.

The system (L, J) is diagrammatically represented by drawing line segments and polygons for the elements of J in which the vertices represent the links incident to the corresponding joints. The transpose system (J, L) is diagrammatically represented by drawing loops for degree zero links, loops with vertex for degree one links, digons (or line segments) for degree two links and polygons for links of degree greater than two. The vertices represent the joints at which the corresponding links are incident to other links (figure 5.5).

The set of coplanar KCs may be given alternative equivalence relations depending on whether the incidence patterns of links at revolute pairs, at joints and at both revolute pairs and joints, are the same. These are called pair equivalence, joint equivalence and equivalence, respectively. Formal definitions are now given.

Two coplanar KCs (L, J, R) and (L_1, R_1, J_1) are pair equivalent if the graphs (L, R) and (L_1, R_1) are isomorphic. They are joint equivalent if there are bijections $\phi : L \rightarrow L_1$ and $\psi : J \rightarrow J_1$ such that $\phi(j) = \psi(j)$ for all $j \in J$. They are equivalent if there are bijections $\phi : L \rightarrow L_1$, $\psi : J \rightarrow J_1$ and $\pi : R \rightarrow R_1$ such that $\phi(r) = \pi(r)$ for all $r \in R$ and $\phi(j) = \psi(j)$ for all $j \in J$ (figure 5.6). If two coplanar KCs are equivalent then they are both pair and joint equivalent, however, the converse does not hold (figure 5.7).

5.2.3 Coplanar KCs under joint equivalence

As noted previously the (L,J,R) description of a coplanar KC considers the links and joints as primary elements and the specification of the revolute pairs in a joint as an ornamentation. Coplanar KCs will be considered in this way in the following development, that is, they will be considered under joint equivalence and the specification of revolute pairs as an ornamentation of the representation.

Coplanar KCs under joint equivalence are specified by the set systems (L,J) or their transpose systems (J,L) . For a coplanar KC described by $K = (L,J)$, let its dual description be denoted by $K^* = (J,L)$. Two coplanar KCs $K = (L,J)$ and $K_1 = (L_1,J_1)$ are considered equivalent if there are bijections $\phi : L \rightarrow L_1$ and $\psi : J \rightarrow J_1$ such that $\phi(j) = \psi(j)$ for all $j \in J$. The systems K and K_1 are isomorphic. If coplanar KCs K and K_1 are equivalent denote this by $K \approx K_1$. Note that $K \approx K_1$ if and only if $K^* \approx K_1^*$.

A sub-KC of a coplanar KC $K = (L,J)$ induced by $L_1 \subseteq L$ is $K[L_1] = (L_1,J_1)$ where $J_1 = \{j \cap L_1 : |j \cap L_1| \geq 2, j \in J\}$. A partial KC of $K = (L,J)$ induced by $J_1 \subseteq J$ is $K[J_1] = (L_1,J_1)$ where $L_1 = \{\ell \in j : j \in J_1\}$. Note that $K[J_1]$ contains no links of degree zero (figure 5.8).

If $K_1 = K[L_1]$ and $K_2 = K[L_2]$ are sub-KCs of $K = (L,J)$ then let $K_1 \cup K_2 = K[L_1 \cup L_2]$. Also if $K_1 = K[J_1]$ and $K_2 = K[J_2]$ are partial KCs of $K = (L,J)$ then let $K_1 \cup K_2 = K[J_1 \cup J_2]$.

A coplanar KC $K = (L,J)$ can be represented by a graph $R(K)$, the representative graph, in which vertices correspond to links

and two vertices are joined by an edge if the corresponding links are incident at a joint. If $E(j) = \{e \in \mathcal{P}_2(j)\}$, $j \in J$ and $E(J) = \{E(j) : j \in J\}$ then $R(K) = (L, E(J))$ (figure 5.9).

If two coplanar KCs $K = (L, J)$ and $K_1 = (L_1, J_1)$ have isomorphic representative graphs $R(K)$ and $R(K_1)$ then K and K_1 are not necessarily equivalent. For instance the coplanar KCs in figure 5.10, shown by their (J, L) diagrams have isomorphic representative graphs but are not equivalent.

Suppose that $R(K)$ and $R(K_1)$ are isomorphic under bijections $\phi : L \rightarrow L_1$ and $\psi : E \rightarrow E_1$ and ϕ does not map the sub-KCs shown in figure 5.10 into one another, then $K \approx K_1$, under the bijections $\phi : L \rightarrow L_1$ and $\hat{\psi} : J \rightarrow J_1$ where $\hat{\psi}(j) = \{l \in \psi(e) : e \in E(j)\}$, $j \in J$.

5.2.4 Coplanar mechanisms

Some ornamentations of coplanar KCs are defined. Coplanar mechanisms are derived from coplanar KCs by fixing a single link. In the definition of a coplanar KC this is represented by rooting or marking a single link as the fixed link. Pair equivalence, joint equivalence and equivalence are defined for coplanar mechanisms as for coplanar KCs but with suitable modifications to preserve the fixed link under the various bijections involved. The ornamentation achieved by designating fixed links in a coplanar KC is kinematic inversion in the sense used by Reuleaux (1876) (figure 5.11). Similar considerations apply to coplanar mechanisms with driven links (figure 5.12). In coplanar mechanisms with only revolute pairs links are often driven by a relative rotation of a link at a pair, thus driven pairs arise (Davies 1968).

5.3 Classification of coplanar KCs

Particular classes of coplanar KCs are now examined in detail. Three criteria for classification are used. First, the degree of freedom, second, the connectedness and third the planarity of the coplanar KCs.

5.3.1 Degree of Freedom

Consider a coplanar KC $K = (L, J)$ with $|L| = p$, $|J| = q$, $|\{\ell \in L : d(\ell) = i\}| = p_i$ and $|\{j \in J : d(j) = i\}| = q_i$. This notation is used throughout for a coplanar KC $K = (L, J)$. The degree of freedom of $K = (L, J)$ is defined by

$$f(K) = 3p - 2 \sum (i-1)q_i - 3, \quad (1)$$

and K is normal if $f(K_1) \geq 0$ for all $K_1 = K[L_1]$, $L_1 \subseteq L$.

The expression for $f(K)$ gives, "in general" the physical degree of freedom of the coplanar mechanisms corresponding to a normal coplanar KC $K = (L, J)$, with a single fixed link. There are special cases which depend on the geometrical properties of the KC, for which the physical degree of freedom is not given by $f(K)$. As Bottema (1950) remarks, "It is well known that mechanisms can be constructed (for which there exist certain metric relations between the bars or for which the contacts are of a certain character) which do not obey the formulae."

A normal coplanar KC $K = (L, J)$ is degenerate if there is a sub-KC $K_1 = K[L_1]$, $L_1 \subset L$, such that $f(K_1) < f(K) \leq |L_1|$. Otherwise it is isokinetic (Crossley 1965). Degeneracy indicates the $f(K)$ independently driven links may not be feasible for certain choices of the links to be driven. These issues are dealt with in more detail by Davies (1968).

A coplanar KC $K = (L, J)$ is irreducible if $f(K) \geq 0$ and $f(K_1) \geq 1$ for all sub-KCs $K_1 = K[L_1]$, $L_1 \subset L$, $|L_1| \geq 2$. A coplanar KC is irreducible in the sense that no proper subset of links (except a single link) induces a sub-KC of degree of freedom zero. Such a degree of freedom zero sub-KC could be replaced by a single link without affecting the global properties of the motion of the KC. Irreducible coplanar KCs with degree of freedom zero are the structures without any substructures. Each normal KC can be derived from an irreducible KC by reversing this replacement.

Degree of freedom, normality and irreducibility are defined naturally for a coplanar KC $K = (L, J, R)$. If K has r revolute pairs, r_i links of pair degree i and $f(K) = f$ then p, q, r, p_i, q_i, r_i and f are related by well known formulae which are derived below. The definitions give

$$\sum p_i = \sum r_i = p, \quad (2)$$

$$\sum q_i = q, \quad (3)$$

$$f = \sum p_i - 2 \sum (i - 1) q_i - 3 = 3p - 2r - 3. \quad (4)$$

Simple counting arguments give

$$\sum i p_i = \sum i q_i = q + r, \quad (5)$$

$$\sum i r_i = 2r. \quad (6)$$

Equations (2) - (6) give

$$f = \sum (3 - 2i) p_i + 2q - 3, \quad (7)$$

$$f = \sum (3 - i) p_i + \sum (2 - i) q_i - 3, \quad (8)$$

$$f = \sum (3 - i) r_i - 3, \quad (9)$$

$$2f = \sum (2 - i) p_i + \sum (4 - i) q_i - 6. \quad (10)$$

In particular, the following inequalities are derived

$$f + 3 \leq 3p_0 + 2p_1 + p_2, \quad (11)$$

$$f + 3 \leq 3r_0 + 2r_1 + r_2, \quad (12)$$

$$2f + 6 \leq 6p_0 + 3p_1 + 2q_2 + q_3. \quad (13)$$

5.3.2 Connectedness

A link-joint path in a coplanar KC $K = (L, J)$ is an ordered set of distinct links and distinct joints $\ell_1, j_1, \ell_2, j_2, \dots, j_k, \ell_{k+1}$, $\ell_i \in L$, $i = 1, 2, \dots, k+1$, and $j_i \in J$, $i = 1, 2, \dots, k$, such that $\ell_i, \ell_{i+1} \in j_i$, $i = 1, 2, \dots, k$. If every two links of K belong to a link-joint path then K is connected.

A k-joint cut in a connected coplanar KC $K = (L, J)$ is $J_1 \subset J$, $|J_1| = k$ such that $K[J - J_1]$ is not connected. A k-link cut is $L_1 \subset L$, $|L_1| = k$ such that $K[L - L_1]$ is not connected. K is k-joint connected if there is no k' -joint cut with $k' < k$ and is k-link connected if there is no k' -link cut with $k' < k$. Note that the presence of degree k links in K does not necessarily imply that K is at most k -joint connected. In particular a normal coplanar KC may be k -joint connected $k \geq 3$ although it necessarily has links of degree one or two by the inequality (11) (figure 5.13).

The definitions of joint and link connectedness can be extended naturally for a coplanar KC (L, J, R) . Further (L, J, R) is k-pair connected if the graph (L, R) is k -edge connected. Also (L, J, R) is k -link connected if and only if the graph (L, R) is k -vertex connected.

The notions of connectedness can be used to refine the

characterization of normal and irreducible KCs. First some definitions are given. Suppose J_0 is a k -joint cut in $K = (L, J)$ and $K[J - J_0]$ has connected components (L_i, J_i) , $i = 1, 2, \dots, n$. Also let $L_i^* = L_i \cup L_0$, where $L_0 = \{\ell : \ell \in j_1 \cap j_2, j_1 \neq j_2 \in J_0\}$, for $k \geq 2$ and $L_i^* = L_i$ for $k = 1$. The $K[L_i^*]$, $i = 1, 2, \dots, n$ are the k -joint components of J_0 (figure 5.14(ii)) and a $K[L_i^*]$ which is not a k' -joint component, $k' < k$, of $J' \subset J_0$ is a proper k -joint component of J_0 (figure 5.14(ii)).

Proposition 5.1. A coplanar KC $K = (L, J)$ is normal if $f(K_1) \geq 0$ for all 2-joint connected sub-KCs $K_1 = K[L_1]$, $L_1 \subseteq L$.

Proof. If there is a sub-KC K_0 with $f(K_0) < 0$ such that no sub-KC of K_0 has this property, then K_0 is either disconnected or has a 1-joint cut. In each case a contradiction arises and thus K must be normal.

Proposition 5.2. A coplanar KC $K = (L, J)$ is irreducible if $f(K_1) \geq 1$ for all 3-joint connected sub-KCs $K_1 = K[L_1]$, $L_1 \subset L$, $|L_1| \geq 2$.

Proof. If there is a sub-KC $K_0 = K[L_0]$, $L_0 \subset L$, $|L_0| \geq 2$ with $f(K_0) \leq 0$ such that no sub-KC of K_0 has this property, then K_0 is either disconnected, has a 1-joint cut or a 2-joint cut. The first two cases provide contradictions. Thus assume K_0 has a 2-joint cut. Let T_1, T_2, \dots, T_n , $n \geq 2$ be the 2-joint components of the 2-joint cut in K_0 . By a counting argument

$$\sum_{i=1}^n f(T_i) \leq f\left(\bigcup_{i=1}^n T_i\right) + (n-1) \leq f(K_0) + (n-1) \leq n-1$$

This provides a contradiction. Thus each sub-KC $K_0 = K[L_0]$, $L_0 \subset L$, $|L_0| \geq 2$ has $f(K_0) \geq 1$. It remains to show that $f(K) \geq 0$.

Suppose $\ell \in L$ is the link of smallest degree in $K = (L, J)$. If $K_0 = K[L - \{\ell\}]$, then $f(K_0) \geq 1$. If $d(\ell) \geq 3$ then K_0 has no links of degree zero or one and at least four links of degree two by inequality (11). This provides a contradiction, thus $d(\ell) \leq 2$. If $d(\ell) = 0, 1$ then the result follows immediately. If $d(\ell) = 2$, then $f(K) = f(K_0) - 1 \geq 0$. Thus the proposition is proved.

5.3.3 Planarity

Finally as a third method of classification, the planarity of coplanar KCs is considered. A connected normal KC $K = (L, J)$ is planar if there is a diagrammatic representation of (J, L) such that polygons, line segments and loops, representing the links, intersect only at their vertices (figure 5.15). This means of classification has a limited use for in the motion of a coplanar KC the links will, in general, "cross over". However, it might be of use in identifying a particularly simple class of coplanar KCs and for those in which only a small range of motion is required. Perhaps the main use is for classifying coplanar KCs with degree of freedom zero, the structures (see section 6.5).

A face in a plane embedding of a coplanar KC is a maximal open set in the plane not intersecting any links. There is also one "infinite" face. Suppose a coplanar KC $K = (L, J)$ has a plane embedding with t_i faces bounded by i links, then Euler's formula gives,

$$a - \sum i p_i + \sum t_i + p = 2 . \quad (14)$$

However,

$$\sum ip_i = \sum iq_i = \sum it_i, \quad (15)$$

and substitution in (14) gives

$$p + q = \sum (i - 1)t_i + 2. \quad (16)$$

The formula for the degree of freedom together with (15) gives

$$\begin{aligned} f &= 3p + 2q - 2 \sum iq_i - 3 \\ &= 3p + 2q - 2 \sum it_i - 3. \end{aligned} \quad (17)$$

Eliminating q from (16) and (17) gives

$$f = p - 2 \sum t_i + 1. \quad (18)$$

5.4 Basic sets for coplanar KCs

The coplanar KCs under joint equivalence can be ornamented by specifying the revolute pairs between links incident at a joint. The coplanar KCs under joint equivalence are specified by a set system (L, J) or its transpose (J, L) . Let A denote the set of connected irreducible coplanar KCs $K = (L, J)$ and \bar{A} the set of connected normal coplanar KCs. Also let $A(p, q, f)$ denote the subset of connected irreducible coplanar KCs $K = (L, J)$ with $|L| = p$, $|J| = q$ and $f(K) = f$. Let $\bar{A}(p, q, f)$ be defined similarly. The aim of this section is to show how the elements in A and \bar{A} can be constructed from basic sets by suitable ornamentation operations.

Consider the set B_1 consisting of connected irreducible coplanar KCs $K = (L, J)$, such that $q_i = 0$, $i \geq 3$. Let $B_1(p, f)$ denote the subset of B_1 consisting of coplanar KCs $K = (L, J)$ with $|L| = p$ and $f(K) = f$. Consider the operation α_1 on the elements of B_1 (figure 5.16), which identifies joints on a link. It preserves the number of links but reduces the number of joints.

The degree of freedom is also preserved. Each element in $A(p,q,f)$ can be constructed from an element in $A(p, q+1, f)$ by applying α_1 . Thus all elements in $A(p,q,f)$ can be constructed from $B_1(p,f)$ by a sequence of several α_1 operations. Note that α_1 also produces coplanar KCs which are not normal or not irreducible from $B_1(p,f)$. If \bar{B}_1 consists of the connected normal coplanar KCs $K = (L,J)$ such that $q_i = 0, i \geq 3$, then elements in \bar{A} may be constructed from \bar{B}_1 , using the operation α_1 .

The other basic set considered is B_2 , which consists of the irreducible coplanar KCs, such that $p_i = 0, i \geq 3$. Elements in A can be obtained from elements in B_2 by a sequence of operations α_2 , which is the "inverse" of the operation α_1 . That is, the expansion of a joint of degree greater than two by increasing the degree of one of its incident links. However, the use of B_2 as a basic set is unsatisfactory since not all elements in A can be constructed in this way (figure 5.17). The set B_2 and the operation α_2 have been used by Mruthyunjaya (1979) to produce the coplanar KCs in B_1 with at most 10 links, although no proof is given of the general application of this method. Manolescu (1979) indicates that a subset of B_2 , namely those with a dyad, and the operation α_2 might be used to construct B_1 . Again no proof is given, but the question is raised as to whether there is a subset of B_2 from which B_1 may be derived. The set \bar{B}_2 is more interesting for if sub-KCs of degree of freedom zero are replaced by single links then all elements in A may be obtained.

Certain subsets of B_1 and B_2 may be considered in their own right as basic sets. Let $B_i(k,k'), i = 1,2$, denote the subsets of

B_i whose elements are k -joint connected and k' -link connected. Note that for $B_1(k, k')$, $k \geq k'$ and for $B_2(k, k')$, $k \leq k'$. Let $B_i(k, k'; p, f)$, $i = 1, 2$, denote the corresponding elements with p links and degree of freedom f . The subsets $\bar{B}_i(k, k'; p, f)$, $i = 1, 2$, are defined similarly. The operation α_1 preserves link connectedness and possibly reduces joint connectedness of elements in A , thus $B_1(k, k'; p, f)$ may be used as a basic set for constructing elements in A which are k -joint connected and k' -link connected.

The elements in $B_i(k, k')$ and $\bar{B}_i(k, k')$, $i = 1, 2$, may have links of degree one. Let $C_i(k, k')$ and $\bar{C}_i(k, k')$, $i = 1, 2$, denote the corresponding subsets without links of degree one. The subsets $C_i(k, k'; p, f)$ and $\bar{C}_i(k, k'; p, f)$, $i = 1, 2$, are also defined. Note that $B_1(k, 2) = C_1(k, 2)$ and an element in B_1 is at most 2-link connected, thus $B_1(k, k')$ is empty for $k' \geq 3$. The set $C_1(k, k'; p, f)$ can be used as a basic set for constructing elements in A which are k -joint connected, k' -link connected, have p links all degree greater than one, and degree of freedom f .

5.5 Conclusion

The irreducible coplanar KCs in $C_1(2, 1)$ correspond to the coplanar KCs, with simple joints and multiple links considered by many authors in their work on the structural analysis of kinematic chains (Crossley 1965, Davies and Crossley 1966, Dobrjanskyj and Freudenstein 1967, Woo 1967). The $C_2(2, 2)$ correspond to the coplanar KCs with simple links and multiple joints. These have been considered in somewhat less detail (Mruthyunjaya 1979). However, both $C_1(2, 1)$ and $C_2(2, 2)$ have been used by Manolescu (1973, 1979) who denotes them by KCs_j and $KCmjsl$, respectively.

It has been noted previously that the presence of links of degree k in a connected coplanar KC K does not necessarily imply that K is at most k -joint connected. This may be considered as a shortcoming in the definition of joint connectedness. The point of view taken here is that the joint cuts corresponding to single links are, in some sense, trivial. However, by considering an alternative definition of joint connectedness these 'trivial' joint cuts may be incorporated.

Suppose that a joint cut in a connected coplanar KC $K = (L, J)$ is defined as a subset of joints $J_1 \subseteq J$, such that the KC $(L, J - J_1)$ is not connected. The single links now correspond to joint cuts. In particular if K has a link of degree one, then the corresponding joint is a 1-joint cut. A corresponding notion of joint connectedness can be defined which is stronger than the original, in the sense that k -joint connectedness in the new definition implies k -joint connectedness in the original definition. The development of the basic sets is somewhat simplified since it is not necessary to exclude the KCs with links of degree one as trivial ornamentations. However, set against this simplification is the fact that under this definition coplanar KCs can be at most 2-joint connected, although it must be admitted that in the following development only 1 and 2-joint connected coplanar KCs are considered. It seems that it is the province of future work to investigate the relative merits of the two approaches and to assess the advantages of allowing joint-connectedness greater than two. It certainly seems that there are advantages for the coplanar KCs in B_2 . Both definitions have their place in the

classification of coplanar KCs, and no doubt both could be used side by side. Future work on these problems would also include a more detailed examination of k-joint components, their definition and relevance to the analysis and synthesis of coplanar KCs.

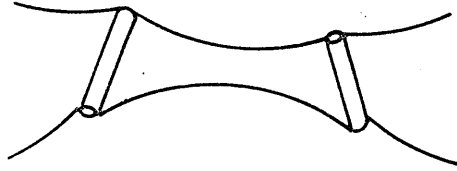


FIGURE 5.1 Two links connected by intermediary link.

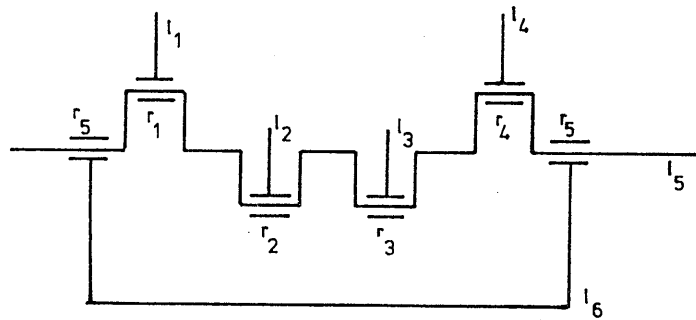


FIGURE 5.2 Schematic diagram of crankshaft and connecting rods in a four cylinder engine.

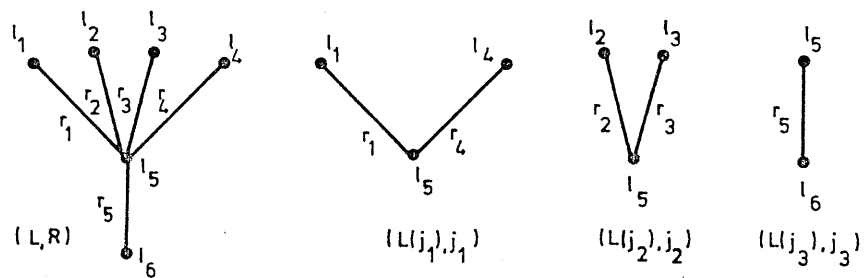


FIGURE 5.3 The graph (L,R) for the kinematic chain in figure 5.2 and subgraphs corresponding to joints in the (L,R,J) description.

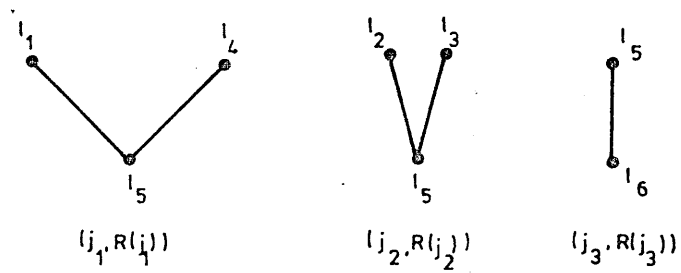


FIGURE 5.4 The subgraphs of the graph (L,R) corresponding to joints in the (L,J,R) description for the kinematic chain in figure 5.2.

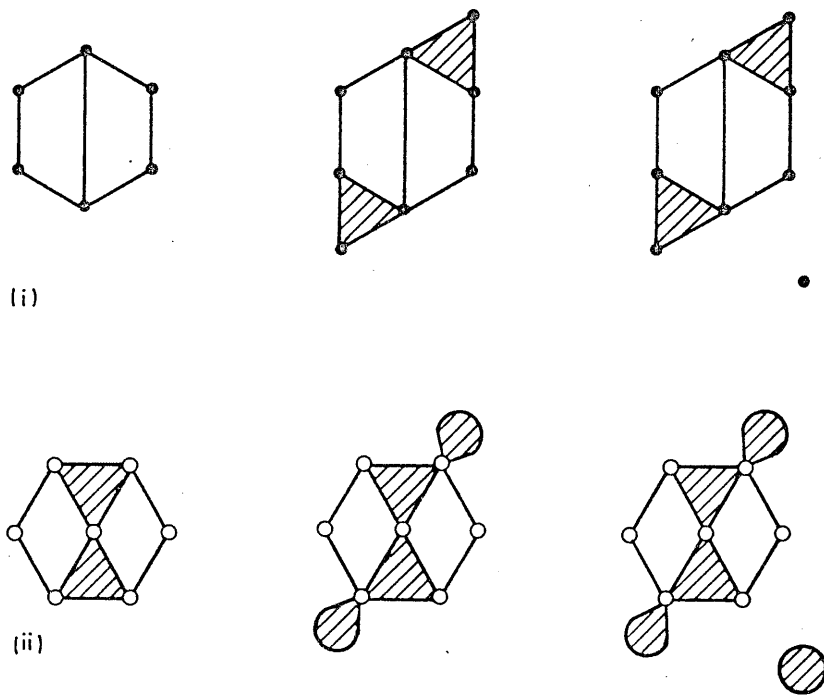
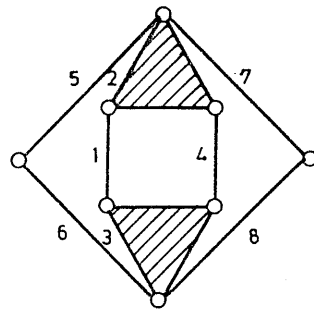
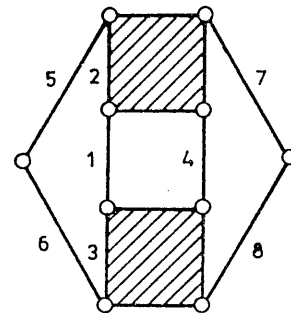


FIGURE 5.5 Diagrammatic representation of coplanar KCs: (i) the systems (L,J) and (ii) the corresponding dual systems (J,L) .



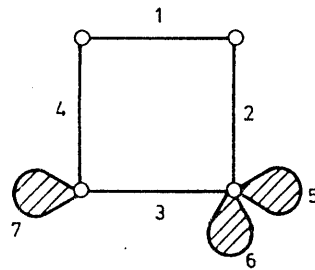
(i)



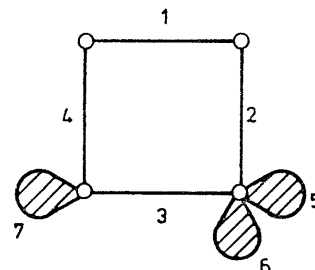
(ii)

$$R = \{(1,2), (1,3), (2,4), (3,4), (2,5), (2,7), (3,6), (3,8), (5,6), (7,8)\}$$

FIGURE 5.6 Pair equivalent coplanar KCs which are not joint equivalent.



(i)



(ii)

$$R = \{(1,2), (2,3), (3,4), (4,1), (4,7), (3,6), (3,5)\} \quad R = \{(1,2), (2,3), (3,4), (4,1), (3,7), (3,6), (2,5)\}$$

FIGURE 5.7 Pair equivalent and joint equivalent coplanar KCs which are not equivalent.

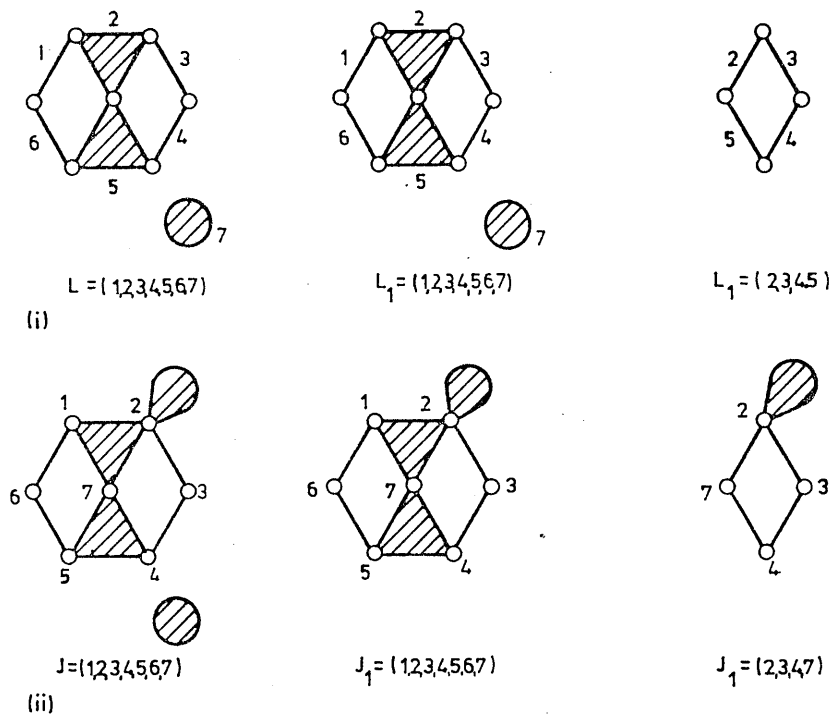


FIGURE 5.8 (i) Sub-KCs and (ii) partial KCs.



FIGURE 5.9 A coplanar KC and its representative graph.



FIGURE 5.10 Non-equivalent coplanar KCs with isomorphic representative graphs.



FIGURE 5.11 Kinematic inversions. The link marked * is fixed.

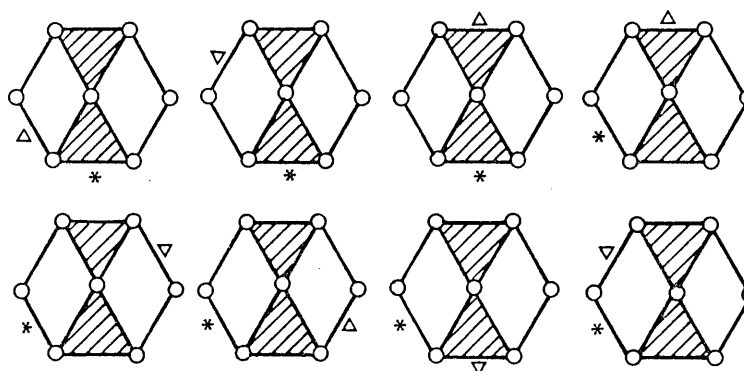
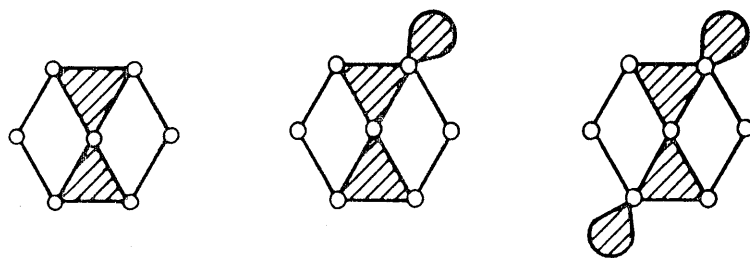
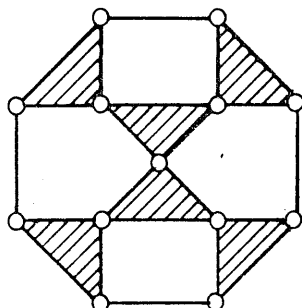


FIGURE 5.12 Driven versions of a coplanar KC. The link marked * is fixed and the link marked Δ is driven.

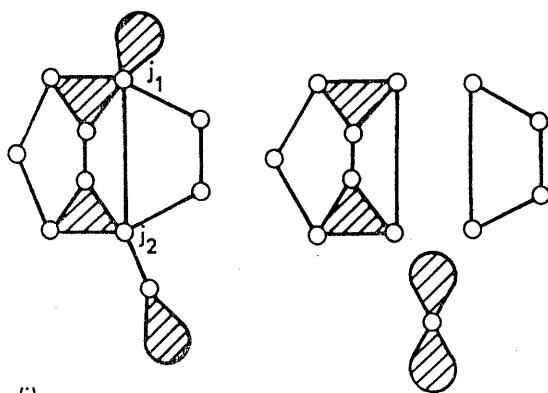


(i)

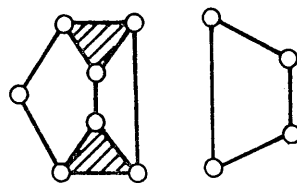


(ii)

FIGURE 5.13 (i) 2-joint connected and (ii) 3-joint connected coplanar KCs.

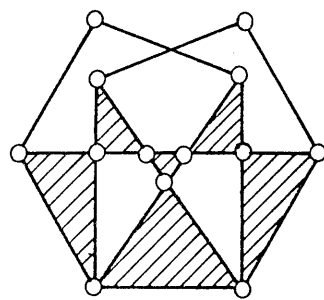


(i)

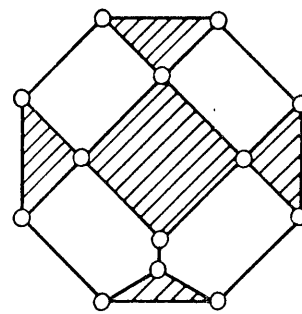


(ii)

FIGURE 5.14 (i) 2-joint components and (ii) proper 2-joint components.



(i)



(ii)

FIGURE 5.15 (i) A coplanar KC which is not planar and (ii) a plane embedding of a coplanar KC which is planar.

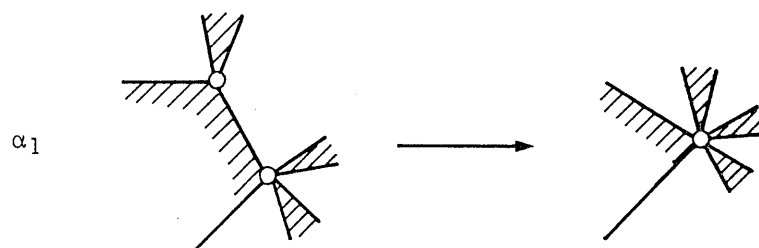


FIGURE 5.16 The operation α_1 .

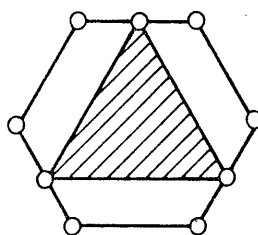


FIGURE 5.17 A coplanar KC not constructible from an element of B_2 by α_2 .

BASIC COPLANAR KINEMATIC CHAINS

6.1 Representation of $C_1(2,1)$ and $C_1(2,2)$

The set system (L,J) for a coplanar KC in B_1 is a simple graph, let $C_1(2,2)$ and $C_1(2,1)$ denote the set systems (L,J) of the corresponding coplanar KCs. For a graph $G = (V,E)$ in C_1

$$f(V_1) = 3|V_1| - 2|E_1| - 3 \quad (1)$$

is defined for $G[V_1]$, $V_1 \subseteq V$ and is the degree of freedom of the sub-KC induced by the links corresponding to V_1 . Two vertices joined by an edge has $f = 1$ and a single vertex has $f = 0$.

A simple graph $G = (V,E)$ is in $C_1(2,1; p,f)$ if and only if

- (1) $|V| = p, |E| = 1/2(3p - f - 3),$
- (2) $f(V_1) \geq 1$ for all subgraphs $G[V_1]$, $V_1 \subseteq V, |V_1| \geq 2.$
- (3) G is 2-edge connected.

A simple graph $G = (V,E)$ is in $C_1(2,2; p,f)$ if and only if (1) and (2) are satisfied, together with

- (4) G is 2-vertex connected.

The graphs in $C_1(2,1; p,f)$ have $p \geq f + 3$ and the lower bound is attained for the cycle graph with $f + 3$ vertices. Also

$$C_1(2,1; p,0) = C_1(2,2; p,0) \text{ and } C_1(2,1; p,1) = C_1(2,2; p,1) .$$

The graphs in $\bar{C}_1(2,1)$ and $\bar{C}_1(2,2)$ are defined similarly but with condition (2) replaced by

- (2) $f(V_1) \geq 0$ for all subgraphs $G[V_1]$, $V_1 \subseteq V.$

6.2 Construction of $C_1(2,1)$ and $C_1(2,2)$

The elements in $C_1(2,1)$ and $C_1(2,2)$ can be constructed in various ways. Four methods will be examined in greater or lesser

detail. Two methods, namely construction according to degree sequences and using Assur groups are reviewed. Two other methods are presented which both seem new. First, the construction by the addition of suspended chains and cycles, and second the construction by using subgraph replacement operations.

6.2.1 Construction by degree sequence

This approach is used by many authors, for example Crossley (1964), Davies and Crossley (1966), Dobrjanskyj and Freudenstein (1967) and Woo (1967). The method consists in finding the solutions p_2, p_3, \dots , to the following equations, for given f and p ,

$$\begin{aligned} f &= 3 \sum p_i - \sum i p_i - 3 = \sum (3 - i) p_i - 3, \\ \sum p_i &= p, \\ \sum i p_i &\equiv 0 \pmod{2}. \end{aligned} \tag{2}$$

Graphs with p_i vertices of degree i are constructed and tested to ensure that conditions (1), (2) and (3) or (4) are satisfied. A modification sometimes employed (Woo 1967) divides the construction into classes corresponding to the number of degree two vertices required. Graphs are constructed without degree two vertices and the appropriate number of such vertices are added by subdividing edges. This operation is effected without examining all edges by using the equivalence of edges under the automorphism group of the graph. This method is also used in Davies and Crossley (1966), where it is expressed in terms of Franke's condensed notation for plane linkages.

There are a number of standard results which might be relevant to these procedures for constructing graphs in $C_1(2,1)$ and $C_1(2,2)$. Suppose that a solution p_2, p_3, \dots , to the equations (2) gives

rise to a potential degree sequence $d_1 \geq d_2 \geq \dots \geq d_p$ with $\sum_{k=1}^p d_k$ even. This is the degree sequence of a 2-edge connected graph (Edmonds 1964) if and only if

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^p \min(k, d_i), \quad (3)$$

$$d_k \geq 2, \quad k = 1, 2, \dots, p.$$

It is the degree sequence of a 2-connected graph if and only if, in addition to (3),

$$d_1 \leq \frac{1}{2}(p+1-f). \quad (4)$$

The inequality (4) implies that the maximum degree of a vertex in a graph in $C_1(2,2; p, f)$ is $\frac{1}{2}(p+1-f)$. There are graphs in $C_1(2,2; p, f)$ with a vertex of degree $\frac{1}{2}(p+1-f)$ (figure 6.1).

Wang and Kleitman (1973) give algorithms to construct n -connected graphs with prescribed degree sequences and it is possible that use can be made of these algorithms in the degree sequence approach to constructing graphs in $C_1(2,2; p, f)$.

6.2.2 Construction by suspended chain and cycle addition

An extension of a result of Whitney (1932) provides a method of construction for graphs in $C_1(2,2)$. This result states that any 2-connected graph can be constructed from a cycle by a sequence of additions of suspended chains or single edges. A suspended chain of length k in a simple graph is a path with k interior vertices each of degree two and distinct endpoints of degree greater than two. Also, if a simple graph $G = (V, E)$ has a suspended chain with interior vertices S then G is said to be obtained from $G-S$ by the addition of the suspended chain. There

is a stronger result for graphs in $C_1(2,2)$ based upon the following proposition.

Proposition 6.1. If $G \in C_1(2,2)$, G not a cycle, then there is a suspended chain with interior vertices S such that $G - S \in C_1(2,2)$.

Proof. Let a suspended chain be denoted by its interior vertices.

Suppose that S_0 is a suspended chain in G , which exists by inequality (11) of section 5.3. Assume that $G - S_0 \notin C_1(2,2)$ for otherwise the proposition holds immediately. Thus $G - S_0$ has two blocks, not single edges, with vertex sets V_1 and W_1 which each contain precisely one cut vertex in $G - S_0$ (figure 6.2). One of the following holds for V_1 :

- (1) $G[V_1]$ is a cycle,
- (2) $G[V_1]$ has a suspended chain S such that there is a suspended chain S_1 in G , $S_1 \subseteq S$, (figure 6.3). This follows since $G[V_1] \in C_1(2,2)$ and by inequality (11) of section 5.3.

If (1) then there is a suspended chain S with vertices in V_1 such that $G - S \in C_1(2,2)$, and the proposition holds. If (2) then either $G - S_1 \in C_1(2,2)$ and the proposition holds, or $G - S_1$ has at least one block, not a single edge, $V_2 \subset V_1$ which contains precisely one cut vertex in $G - S_1$. The argument can be repeated for V_2 in place of V_1 . However, it cannot be repeated indefinitely, thus the required suspended chain exists and the proposition holds.

Corollary 6.1.1. If $G \in C_1(2,2)$, G not a cycle, then G contains two suspended chains with interior vertices $S_1 \neq S_2$ such that $G - S_i \in C_1(2,2)$, $i = 1, 2$.

Proof. The result is a corollary of the proof of proposition 6.1, since G contains at least two suspended chains.

Corollary 6.1.2. If $G \in C_1(2,2)$, G not a cycle, then G can be constructed from a cycle by a sequence of additions of suspended chains (figure 6.4(i)).

A suspended cycle of length k in a simple graph is a cycle with $k + 1$ vertices of which exactly k are degree two. There is a result similar to corollary 6.1.1 for graphs in $C_1(2,1)$ given in the following proposition.

Proposition 6.2. If $G \in C_1(2,1)$, G not a cycle, then G contains two suspended chains or cycles with interior vertices $S_1 \neq S_2$ such that $G - S_i \in C_1(2,1)$, $i = 1, 2$.

Proof. It may be assumed that $G \notin C_1(2,2)$. Let G contain blocks with vertices $B(1), B(2), \dots, B(k)$ such that the $B(i)$, $i = 1, 2, \dots, k$ contain precisely one cut vertex in G . Since $k \geq 2$, corollary 6.1.1 provides the result.

Corollary 6.2.1. If $G \in C_1(2,1)$, G not a cycle, then G can be constructed from a cycle by a sequence of suspended chain and suspended cycle additions.

Suppose that the construction of a graph in $C_1(2,2; p, f)$ from an n -cycle requires a_i additions of suspended chains of length i , then

$$p = n + \sum i a_i , \quad (5)$$

$$\frac{1}{2}(3p - f - 3) = n + \sum (i + 1) a_i . \quad (6)$$

(5) and (6) give

$$\sum a_i = \frac{1}{2}(p - f - 3) . \quad (7)$$

Therefore given p and f , partitions of $\frac{1}{2}(p - f - 3)$ correspond to possible a_i . The order in which the suspended chains can be added is constrained to ensure that the irreducibility condition is not necessarily violated. The particular vertices at which the suspended chains are added can also be examined. The construction of the graph in figure 6.4(i) has $a_1 = 2$, $a_3 = 2$ and $\frac{1}{2}(p - f - 3) = 4$. An alternative construction is shown in figure 6.4(ii) with $a_1 = 1$, $a_2 = 2$, $a_3 = 1$.

Similar considerations apply to constructing a graph in $C_1(2,1; p, f)$ from an n -cycle by a_i and b_i additions of suspended chains and cycles, respectively, of length i . In this case

$$p = n + \sum i(a_i + b_i) , \quad (8)$$

$$\frac{1}{2}(3p - f - 3) = n + \sum (i + 1)(a_i + b_i) . \quad (9)$$

(8) and (9) give

$$\sum (a_i + b_i) = \frac{1}{2}(p - f - 3) . \quad (10)$$

Figure 6.5 shows a graph in $C_1(2,2; 9,0)$ which can be constructed using both suspended chain and suspended cycle additions with $a_1 = 1$, $b_2 = 1$, $b_3 = 1$ and $1/2(p - f - 3) = 3$.

6.2.3 Construction by subgraph replacement

The graphs in $C_1(2,1; p,f)$ are now examined in more detail. Properties are given which relate to subgraphs and to the distribution of degree two vertices.

Proposition 6.3. If $G = (V,E) \in C_1(2,1)$, $V_1, V_2, \dots, V_m \subseteq V$ and $G[V_1 \cup V_2 \cup \dots \cup V_m]$ has n edges not in $G[V_1], G[V_2], \dots, G[V_m]$, then

$$\begin{aligned} f\left(\bigcup_{i=1}^m V_i\right) &= \sum_{i=1}^m f(V_i) - \sum_{i>j} f(V_i \cap V_j) + \sum_{i>j>k} f(V_i \cap V_j \cap V_k) \\ &\quad + \dots + (-1)^{m+1} f\left(\bigcap_{i=1}^m V_i\right) - 2n. \end{aligned} \quad (11)$$

Proof. Use a straightforward counting argument.

The next propositions are concerned with particular subgraphs of graphs in $C_1(2,1)$. If $G = (V,E) \in C_1(2,1)$ let $P(u,v)$ and $Q(u,v)$ denote subsets of V containing $u, v \in V$ such that $f(P(u,v)) = 1$, $f(Q(u,v)) = 2$ and there are no subsets of $P(u,v)$ and $Q(u,v)$ with the same properties. Figure 6.6 shows examples of the sets $P(u,v)$ and $Q(u,v)$ for a graph in $C_1(2,1; 12,1)$. The subgraphs $P(u,v)$ and $Q(u,v)$ correspond to subchains connecting the links corresponding to u and v which allow one and two degrees of relative freedom, respectively, between the two links.

Proposition 6.4. If $G = (V,E) \in C_1(2,1; p,f)$, $u, v \in V$, then

$P(u,v)$, if it exists, is unique except when $V = P_1(u,v) \cup P_2(u,v)$ and $f(V) = 0$.

Proof. Suppose there are $P_1(u,v) \neq P_2(u,v)$ then proposition 6.3 with $m = 2$ provides a contradiction unless $V = P_1(u,v) \cup P_2(u,v)$ and $f(V) = 0$.

Proposition 6.5. If $G = (V,E) \in C_1(2,1; p,f)$, $u,v \in V$ and there is $P(u,v) \neq \{u,v\}$ then each $Q(u,v) \subset P(u,v)$ unless $V = Q(u,v) \cup P(u,v)$, $f(V) = 0$.

Proof. As $P(u,v) \neq \{u,v\}$ then $G[P(u,v)]$ contains a degree two vertex $w \neq u,v$. Thus there is $Q(u,v) \subseteq P(u,v) - \{w\}$. Suppose there is $Q_1(u,v) \neq P(u,v)$, then $f(Q_1(u,v) \cap P(u,v)) \geq 3$ for otherwise definitions of $P(u,v)$, $Q(u,v)$ and $Q_1(u,v)$ are contradicted. By proposition 6.3 there is a contradiction unless $V = Q_1(u,v) \cup P(u,v)$, $f(V) = 0$.

Proposition 6.6. If $G = (V,E) \in C_1(2,1;p,f)$, $u, v \in V$, $f \geq 1$, $Q_1(u,v) \neq Q_2(u,v)$ then $Q_1(u,v)$ and $Q_2(u,v)$ are unique and $P(u,v) = Q_1(u,v) \cup Q_2(u,v)$ unless $(u,v) \in E(G)$.

Proof. Let $Q_1 = Q_1(u,v)$, $Q_2 = Q_2(u,v)$. By proposition 6.3 $f(Q_1 \cap Q_2) + f(Q_1 \cup Q_2) \leq 4$ and by definitions $f(Q_1 \cap Q_2) = 3$. Thus $f(Q_1 \cup Q_2) = 1$. Suppose there is $Q_3 = Q_3(u,v) \neq Q_1(u,v)$, $Q_2(u,v)$ then $Q_3 \subset Q_1 \cup Q_2$ and similarly $Q_1 \subset Q_2 \cup Q_3$, $Q_2 \subset Q_1 \cup Q_3$. This provides a contradiction. If $(u,v) \notin E(G)$ then $P(u,v) \subseteq Q_1 \cup Q_2$ and by proposition 6.5 equality holds.

The next propositions are concerned with the distribution of degree two vertices in the graphs belonging to $C_1(2,1)$.

Proposition 6.7. If $G = (V, E) \in C_1(2, 1; p, f)$ then each suspended chain has length $k \leq f + 1$ and $p_2 \geq f + 3$. Also $p_2 = f + 3$ if and only if $p_i = 0, i \geq 4$.

Proof. Follows immediately from proposition 6.3 and inequality (11) in section 5.3.

Proposition 6.8. If $G = (V, E) \in C_1(2, 1)$ then $d(v) \leq p_2$ for all $v \in V$. Also if $d(v) = p_2$ for some $v \in V$ then $f(V) = 0$ and $d(w) \leq 3$ for all $w \in V - \{v\}$.

Proof. The result follows from the equation (8) of section 5.3, namely

$$p_2 = f(V) + 3 + \sum_{i \geq 4} (i - 3)p_i.$$

Corollary 6.8.1. If $G = (V, E) \in C_1(2, 1)$ has all degree two vertices adjacent to a single vertex $v \in V$, then $d(v) = p_2$, $f(V) = 0$ and $d(w) \leq 3$ for all $w \in V - \{v\}$ (figure 6.7).

Let degree two vertices be called remote if they are not adjacent to a common vertex. Consider the graphs in figure 6.8, which have no remote degree two vertices. They are the only graphs in $C_1(2, 1)$ with $f \geq 1$ and no remote degree two vertices.

Consider the subgraph $G - V_1$, of $G = (V, E) \in C_1(2, 1)$ where V_1 is the set of degree two vertices in V . Now consider the vertices in G of degree greater than two and adjacent to exactly zero, one and two vertices of degree greater than two. Denote these sets of vertices by $V(0)$, $V(1)$ and $V(2)$ respectively. Suppose there are s_i and c_i suspended chains and cycles, respectively, of length i , in G . If G is not a cycle then by inequality (11) of section 5.3, applied to $G - V_1$,

$$3|V(0)| + 2|V(1)| + |V(2)| \geq f(V) + 3 + \sum_{i=1}^{f+1} (2-i)s_i + \sum_{i=1}^{f+1} (1-i)c_i. \quad (12).$$

Proposition 6.9. If $G \in C_1(2,1; p, f)$, G not a cycle, $c_i = s_i = 0$, $i \geq 3$, then there is $v \in V(0) \cup V(1) \cup V(2)$ such that if $v \in V(0)$, $d(v) < 6$, if $v \in V(1)$, $d(v) < 5$ and if $v \in V(2)$ then $d(v) < 4$.

Proof. Suppose there is no such vertex then

$$6|V(0)| + 4|V(1)| + 2|V(2)| \leq 2p_2$$

which contradicts inequality (12) above.

Proposition 6.10. If $G \in C_1(2,1; p, f)$, G not a cycle, $c_i = s_i = 0$, $i \geq 3$, such that $d(v) \geq 6$, $v \in V(0)$ and $d(v) \geq 5$, $v \in V(1)$, then there is a vertex of degree two adjacent to $v_1 \in V(2)$, $d(v_1) = 3$ and $v_2 \in V(0) \cup V(1) \cup V(2)$.

Proof. Suppose there are k vertices of degree three in $V(2)$ and there is no vertex of degree two as specified then

$$6|V(0)| + 4|V(1)| + 2(|V(2)| - k) + k \leq 2p_2 - k,$$

which contradicts inequality (12) above.

Proposition 6.11. If $G \in C_1(2,1; p, f)$, G not a cycle, $c_i = s_i = 0$, $i \geq 3$, then there is a vertex of degree two in G adjacent to $v_1, v_2 \in V(0) \cup V(1) \cup V(2)$.

Proof. Suppose there is no such vertex then

$$3|V(0)| + 2|V(1)| + |V(2)| \leq p_2$$

which contradicts inequality (12) above.

It seems feasible that the knowledge about the structure of graphs in $C_1(2,1)$ given in propositions 6.9, 6.10 and 6.11 may be

used to construct these graphs by subgraph replacement operations. However, the number of operations required and the complexity of their application would seem to rule out their use. Proposition 6.12 will give a relatively simple method of construction.

Consider the subgraph replacement operations β_1 , β_2 and β_3 shown in figure 6.9. There are variants of the operations depending on the relative orientations of the subgraphs on right and left. For instance β_1 has $\{w_1, w_2, w_3, w_4\} = \{v_1, v_2, v_3, v_4\}$ but w_i is not necessarily the same as v_i . The operations β_2 and β_3 may also be considered as variants of β_1 under the identification of vertices.

Proposition 6.12. If $G \in C_1(2,1; p, f)$, $f \geq 1$ then G can be constructed from a graph in $C_1(2,1; p-2, f)$ by the operation β_1, β_2 , or β_3 .

Proof. Note first that the proposition does not hold for $f = 0$ (see figure 6.10). If G is a graph in figure 6.8 then β_3 constructs G . Thus assume G contains the configuration in figure 6.11, where v_1, v_2, v_3 and v_4 are distinct vertices. The degree two vertices u_1 and u_2 are remote.

Let u_1 and u_2 belong to suspended chains or cycles S_1 and S_2 respectively. If $G-S_1$ or $G-S_2$ is 1-edge separated then by proof of propositions 6.1 and 6.2 there is a pair of remote degree two vertices such that β_1 constructs G . Thus suppose that $G-S_1$ and $G-S_2$ are in $C_1(2,1)$. If $G-(S_1 \cup S_2)$ is 1-edge separated then β_1 constructs G . Thus suppose that $G-(S_1 \cup S_2) \in C_1(2,1)$. If $|S_1| \geq 2$ then β_1 constructs G ; similarly for S_2 . Thus suppose that $G-\{u_1, u_2\} \in C_1(2,1)$ for each pair $\{u_1, u_2\}$ of remote degree two vertices.

If there is no $Q(v_i, v_j)$ for some $i \neq j \in \{1, 2, 3, 4\}$ and $(v_i, v_j) \notin E(G)$ then β_1 constructs G . Suppose there is a $Q(v_i, v_j)$ for all $i \neq j \in \{1, 2, 3, 4\}$. Let $Q_1 = Q(v_1, v_2)$, $Q_2 = Q(v_1, v_3)$, $Q_3 = Q(v_2, v_3)$, $Q_4 = Q(v_2, v_4)$, $Q_5 = Q(v_3, v_4)$, $Q_6 = Q(v_1, v_4)$ and let $Q_{ij} = Q_i \cup Q_j$, $Q_{ijk} = Q_i \cup Q_j \cup Q_k$ and so on. Also let $R(u_1, u_2) = Q_1 \cup Q_2 \cup \dots \cup Q_6 \cup \{u_1\} \cup \{u_2\}$.

If $u_1 \in Q_1$ or $u_2 \in Q_5$ then β_1 constructs G . If $u_1 \in Q_i$, $i = 2, 3, 4, 6$, then β_1 constructs G unless there is $Q_i' \neq Q_i$, $u_1 \notin Q_i'$; similarly for u_2 . Thus suppose $u_1, u_2 \notin Q_i$, $i = 1, 2, 3, 4, 5, 6$.

By equation (11) in proposition 6.3

$$2 \leq f(Q_{123}) \leq 6 - (f(Q_1 \cap Q_2) + f(Q_2 \cap Q_3) + f(Q_3 \cap Q_1)) + f(Q_1 \cap Q_2 \cap Q_3), \quad (13)$$

$$2 \leq f((Q_i \cap Q_j) \cup (Q_i \cap Q_k)) \leq f(Q_i \cap Q_j) + f(Q_i \cap Q_k) - f(Q_1 \cap Q_2 \cap Q_3) \quad i \neq j \neq k \in \{1, 2, 3\}. \quad (14)$$

Equations (13 and (14) give

$$2 \leq f(Q_{123}) \leq 3 - 1/2 f(Q_1 \cap Q_2 \cap Q_3), \quad (15)$$

$$2 \leq f(Q_{123}) \leq 3. \quad (16)$$

Consider the two cases, $f(Q_{123}) = 2, 3$. If $f(Q_{123}) = 2$ then $v_4 \notin Q_{123}$ and

$$3 \leq f(Q_{12345}) \leq 6 - (f(Q_{123} \cap Q_4) + f(Q_4 \cap Q_5) + f(Q_5 \cap Q_{123})) + f(Q_{123} \cap Q_4 \cap Q_5), \quad (17)$$

$$1 \leq f((Q_{123} \cap Q_4) \cup (Q_{123} \cap Q_5)) \leq f(Q_{123} \cap Q_4) + f(Q_{123} \cap Q_5) - f(Q_{123} \cap Q_4 \cap Q_5), \quad (18)$$

$$2 \leq f(Q_i \cap Q_j) \cup (Q_{123} \cap Q_i) \leq f(Q_i \cap Q_j) + f(Q_i \cap Q_{123}) - f(Q_{123} \cap Q_i \cap Q_j), \quad i \neq j \in \{4,5\}. \quad (19)$$

Equations (17), (18) and (19) give

$$5 \leq 2(f(Q_{123} \cap Q_4) + f(Q_{123} \cap Q_5) + f(Q_4 \cap Q_5)) - 3f(Q_{123} \cap Q_4 \cap Q_5), \quad (20)$$

$$3 \leq f(Q_{12345}) \leq 7/2 - 1/2 f(Q_{123} \cap Q_4 \cap Q_5). \quad (21)$$

Thus $f(Q_{12345}) = 3$, $f(Q_{123} \cap Q_4 \cap Q_5) \leq 1$ and $Q_6 \subset Q_{12345}$.

If $Q_{123} \cap Q_4 \cap Q_5 = \emptyset$ then $f(Q_{123} \cap Q_4) = f(Q_{123} \cap Q_5) = f(Q_4 \cap Q_5) = 0$. Thus $Q_{123} \cap Q_4 = v_2$, $Q_{123} \cap Q_5 = v_3$, $Q_4 \cap Q_5 = v_4$ which yields a contradiction since $Q_6 \subset Q_{12345}$.

If $Q_{123} \cap Q_4 \cap Q_5 = v \in V(G)$ then $f(Q_{123} \cap Q_4 \cap Q_5) \geq 1$ since $v_4 \notin Q_{123}$. If $v = v_2$, $f(Q_{123} \cap Q_4 \cap Q_6) \leq f(Q_{123} \cap Q_4) + f(Q_6) - f((Q_{123} \cap Q_4) \cup Q_6)$, $f(Q_{123} \cap Q_4) = 1$, $f((Q_{123} \cap Q_4) \cup Q_6) \geq 3$, thus $f(Q_{123} \cap Q_4 \cap Q_6) = 0$ and $R(u_1, u_2)$ is configuration (c) shown in figure 6.12. Similarly if $v = v_3$. If $v \neq v_2, v_3$ then $f(Q_{123} \cap Q_4) = 1 = f(Q_{123} \cap Q_5)$ and $f(Q_{12345}) = 3$. As $Q_6 \subset Q_{12345}$, $R(u_1, u_2)$ is the configuration (c) or (d) in figure 6.12.

If $f(Q_{123} \cap Q_4 \cap Q_5) = 1$, then $f(Q_{123} \cap Q_5) = f(Q_{123} \cap Q_4) = 1$ and $f((Q_{123} \cap Q_4) \cup (Q_{123} \cap Q_5)) = 1 = f(Q_{123} \cap Q_{45})$. Also $f(Q_{123} \cap Q_{45} \cap Q_6) \leq f(Q_{123} \cap Q_{45}) + f(Q_6) - f((Q_{123} \cap Q_{45}) \cup Q_6)$ and $f((Q_{123} \cap Q_{45}) \cup Q_6) \geq 3$. Thus $f(Q_{123} \cap Q_{45} \cap Q_6) = 0$ and $R(u_1, u_2)$ is the configuration (c) in figure 6.12.

If $f(Q_{123}) = 3$, then $f(Q_1 \cap Q_2 \cap Q_3) \leq 0$. If $f(Q_1 \cap Q_2 \cap Q_3) = 3$ then $f(Q_1 \cap Q_2) = f(Q_2 \cap Q_3) = f(Q_3 \cap Q_1) = 1$ and thus $Q_1 \cap Q_2 = v_1$,

$Q_2 \cap Q_3 = v_3$, $Q_3 \cap Q_1 = v_2$. If $f(Q_1 \cap Q_2 \cap Q_3) = 0$ then $f(Q_1 \cap Q_2) = f(Q_2 \cap Q_3) = f(Q_3 \cap Q_1) = 1$. As a consequence $R(u_1, u_2)$ is the configuration (a) or (b) in figure 6.12.

Note that as a consequence of the above argument if $(v_i, v_j) \in E(G)$ then there is no $Q(v_k, v_\ell)$ for some $k \neq \ell \in \{1, 2, 3, 4, 5, 6\}$. Thus suppose that each pair $\{u_1, u_2\}$ of remote degree two vertices gives rise to $R(u_1, u_2)$ in one of the configurations (a), (b), (c) or (d) in figure 6.12.

Consider a pair $\{u_1, u_2\}$ of remote degree two vertices. If $R(u_1, u_2)$ contains a $u_3 \neq u_1, u_2$, $d(u_3) = 2$, remote from u_1 and u_2 then either there is no $R(u_1, u_3)$ or the minimality of some $Q(v_i, v_j) \subset R(u_1, u_2)$ is contradicted. If $R(u_1, u_2)$ contains a $u_3 \neq u_1, u_2$, $d(u_3) = 2$, not remote from u_1 or u_2 then apply β_2 or β_3 to construct G . Thus suppose $R(u_1, u_2)$ does not contain a $u_3 \neq u_1, u_2$, $d(u_3) = 2$.

Consider all pairs of remote degree two vertices. If $R(u_1, u_2)$ is type (c) or (d) for each pair $\{u_1, u_2\}$ of remote degree two vertices then there is a vertex of degree greater than p_2 , which by proposition 6.8 is a contradiction. Thus suppose there is a pair $\{u_1, u_2\}$ of remote degree two vertices such that $R(u_1, u_2)$ is type (a) or (b).

If $R(u_1, u_2)$ is type (a) or (b) then there is a $u_3 \neq u_1, u_2$, $d(u_3) = 2$, remote from u_1 or u_2 . If u_3 is remote from u_1 or u_2 then either $R(u_1, u_3)$ or $R(u_2, u_3)$ is type (a) or (b) for if not then $R(u_1, u_2)$ would be type (c) or (d). If $R(u_1, u_3)$ is type (a) or (b), then $f(R(u_1, u_3) \cup R(u_1, u_2)) = 1$ and $R(u_2, u_3) \subseteq R(u_1, u_3) \cup R(u_1, u_2)$, which is impossible. If u_3 is remote from u_1 but not u_2 , then

β_2 constructs G; similarly for u_3 remote from u_2 but not u_1 . The proposition is thus established.

Figure 6.13 shows the construction of coplanar KCs using the operations β_1 , β_2 , and β_3 .

6.2.4 Construction by Assur groups

Adequate coverage of this idea does not seem to have appeared in the english literature on the subject (see Dobrjanskyj 1966, Chapter 2, sections 4 and 5) thus the explanation is tentative and is included for the sake of completeness.

The idea underlying Assur group construction is that a given coplanar KC (indeed any KC) can be constructed by the addition of certain sets of coplanar KCs to a smaller coplanar KC with the same degree of freedom. This addition is effected by specifying joints of attachment. In order to preserve the degree of freedom the added coplanar KCs have the property that if all the joints of attachment were connected to a single link the result would have zero degree of freedom. Also no subset of the added coplanar KC has this property. The way in which a coplanar KC is constructed from a single fixed link and a set of cranks serves to provide a scheme of classification (figure 6.14). The theoretical basis of this method requires that each coplanar KC has a sub-KC of the correct type. The implementation of the method seems to require a set of coplanar KCs for addition, whose size increases with the number of links in the KCs to be constructed.

The construction of the sets of coplanar KCs required for

addition is clearly associated with the generation of elements in $C_1(1,2; p,0)$, $p = 3, 5, 7, \dots$ but with more complex subgraph conditions (figure 6.15).

6.3 Representation of $C_2(1,2)$, $C_2(2,2)$, $\bar{C}_2(1,2)$, $\bar{C}_2(2,2)$

The transpose (J,L) for a coplanar $KC(L,J)$ in B_2 is a simple graph. In this section $C_2(2,2)$ and $C_2(1,2)$ denote these graphs. For a graph $G = (V,E) \in C_2$

$$f(V_1) = 2|V_1| - |E_1| - 3 \quad (22)$$

is defined for $G[V_1]$, $V_1 \subseteq V$. A single vertex has $f = -1$, two vertices joined by an edge has $f = 0$ and three vertices joined by two edges has $f = 1$. Also, two non-adjacent vertices have $f = 1$.

A simple graph $G = (V,E)$ is in $C_2(1,2; p,f)$ if and only if

- (1) $|E| = p$, $|V| = 1/2(p + f + 3)$,
- (2) $f(V_1) \geq 1$ for all 3-vertex connected subgraphs $G[V_1]$, $V_1 \subset V$,
- (3) G is 2-edge connected.

A simple graph is in $C_2(2,2; p,f)$ if and only if (1) and (2) above are satisfied together with,

- (4) G is 2-vertex connected.

The graphs in $C_2(1,2; p,f)$ have $p \geq f + 3$ and the lower bound is attained for the cycle graph with $f + 3$ vertices. Also $C_2(1,2; p,1) = C_2(2,2; p,1)$ and $C_2(1,2; p,0) = C_2(2,2; p,0)$.

The graphs in $\bar{C}_2(1,2)$ and $\bar{C}_2(2,2)$ are defined similarly but with condition (2) replaced by

- (2) $f(V_1) \geq 0$ for all 2-vertex connected subgraphs $G[V_1]$, $V_1 \subseteq V$.

6.4 Construction of $C_2(1,2)$, $C_2(2,2)$, $\bar{C}_2(1,2)$, $\bar{C}_2(2,2)$

The graphs in $C_2(1,2)$ and $C_2(2,2)$ can be constructed by degree sequence methods similar to those in section 6.2. They may also be constructed by the addition of suspended chains, suspended cycles and single edges, using Whitney's theorem, (Whitney 1932), on the construction of 2-vertex connected graphs. Consider a single edge as a suspended chain of length zero.

Suppose that the construction of a graph in $C_2(1,2; p, f)$ from an n -cycle requires a_i additions of suspended chains of length i and b_i additions of suspended cycles of length i . Counting edges and vertices gives

$$p = n + \sum (i+1)a_i + \sum (i+1)b_i, \quad (23)$$

$$1/2(f + p + 3) = n + \sum ia_i + \sum ib_i. \quad (24)$$

(23 and (24) give

$$\sum (a_i + b_i) = 1/2(p - f - 3). \quad (25)$$

In this section methods based on subgraph replacement are examined in detail. First some properties of graphs in $\bar{C}_2(1,2)$ and $\bar{C}_2(2,2)$ are given. Note that these properties also hold for graphs in $C_2(1,2)$ and $C_2(2,2)$, respectively.

Proposition 6.13. If $G = (V, E) \in \bar{C}_2(1,2)$, $v_1, v_2, \dots, v_m \subseteq V$ and $G[v_1 \cup v_2 \cup \dots \cup v_m]$ has n edges not in $G[v_1]$, $G[v_2]$, \dots , $G[v_m]$ then

$$\begin{aligned} f\left(\bigcup_{i=1}^m v_i\right) &= \sum_{i=1}^m f(v_i) - \sum_{i>j} f(v_i \cap v_j) + \sum_{i>j>k} f(v_i \cap v_j \cap v_k) \\ &\quad + \dots + (-1)^{m+1} f\left(\bigcap_{i=1}^m v_i\right) - n. \end{aligned} \quad (26)$$

Proof. Straightforward counting argument.

Proposition 6.13 has some simple consequences for the connectedness of graphs in $\overline{C}_2(1,2; p,0)$ and $\overline{C}_2(2,2; p,1)$. First $\overline{C}_2(1,2; p,0) = \overline{C}_2(3,3; p,0)$ and second $\overline{C}_2(1,2; p,1) = \overline{C}_2(2,2; p,1)$.

Consider $G \in \overline{C}_2(2,2)$. Let $T(G)$ denote the set of vertices $v \in V(G)$, $d(v) = 3$, such that $G - e \in \overline{C}_2(2,2)$ for some edge e incident to v . Let $S(G)$ denote the set of suspended chains in G , such that for a suspended chain $S \in S(G)$, $G - S \in \overline{C}_2(2,2)$, where S denotes the interior vertices of the suspended chain.

Proposition 6.14. If $G \in \overline{C}_2(2,2)$, G not a cycle, then

$$|T(G)| + |S(G)| \geq 2.$$

Proof. Extensive use is made of inequality (13) of section 5.3 which implies that for any graph in $\overline{C}_2(2,2)$ there are at least three degree two or degree three vertices.

Suppose first that there is $v_0 \in V(G)$, $d(v_0) = 3$, $v_0 \notin T(G)$. Assume that $G - v_0$ has at least two blocks, not single edges, with vertex sets V_1 and W_1 , such that V_1 and W_1 each contain exactly one cut vertex in $G - v_0$. Otherwise $|S(G)| \geq 2$ and the proposition holds. One of the following holds for V_1 :

- (1) $G[V_1]$ is a cycle,
- (2) $G[V_1]$ has a suspended chain S such that there is a suspended chain S_1 in G , $S_1 \subseteq S$, (this follows since $G[V_1] \in \overline{C}_2(2,2)$),
- (3) $G[V_1]$ has a vertex v_1 , $d(v_1) = 3$ in G and $G[V_1]$.

If (1) then $|S(G)| \geq 1$. If (2) then either $S_1 \in S(G)$

and $|S(G)| \geq 1$ or $G - S_1$ has at least one block, not a single edge, with vertex set $V_2 \subset V_1$, which contains exactly one cut vertex in $G - S_1$. If (3) then either $v_1 \in T(G)$ and $|T(G)| \geq 1$ or $G - v_1$ has at least one block, not a single edge, with vertex set $V_2 \subset V_1$, which contains precisely one cut vertex in $G - v_1$. The above argument can be repeated, in each case, with V_2 in place of V_1 . However, it cannot be repeated indefinitely since $V_2 \subset V_1$. Thus $|T(G)| + |S(G)| \geq 1$, for V_1 . Similarly for W_1 , thus $|T(G)| + |S(G)| \geq 2$ and the proposition holds.

Second, suppose that there is a suspended chain $S_0 \notin S(G)$. The graph $G - S_0$ has at least two blocks, not single edges, with vertex sets V_1 and W_1 , such that V_1 and W_1 each contain exactly one cut vertex in $G - S_0$. The above argument can be applied to V_1 and W_1 yielding $|T(G)| + |S(G)| \geq 2$.

Finally suppose that each suspended chain is in $S(G)$ and each degree three vertex is in $T(G)$. If $|T(G)| = 0$ then there is at least one suspended chain in G . If there is only one suspended chain $S \in S(G)$ then inequality (13) of section 5.3 for $G - S \in \overline{C}_2(2,2)$ is contradicted. Thus $|S(G)| \geq 2$. If $|T(G)| = 1$ then there is at least one suspended chain $S \in S(G)$, thus $|T(G)| + |S(G)| \geq 2$.

Now consider $G \in \overline{C}_2(1,2)$. Let $T(G)$ denote the set of vertices $v \in V(G)$, $d(v) = 3$, such that $G - e \in \overline{C}_2(1,2)$ for some edge e incident to v . Let $S(G)$ denote the set of suspended chains and suspended cycles in G such that for a suspended chain with interior vertices S , in $S(G)$, $G - S \in \overline{C}_2(1,2)$.

Proposition 6.15. If $G \in \overline{C}_2(1,2)$, G not a cycle, then

$$|T(G)| + |S(G)| \geq 2.$$

Proof. If $G \in \overline{C}_2(2,2)$ then the proposition holds by proposition

6.14. If $G \in \overline{C}_2(1,2)$ then there are at least two blocks with interior vertices V_1 and W_1 in G which contain exactly one cut vertex in G . Consider V_1 . If V_1 is a cycle then $|S(G)| \geq 1$ and if V_1 is not a cycle then by proposition 6.14 $|T(G)| + |S(G)| \geq 1$.

A similar argument applies to W_1 , thus $|T(G)| + |S(G)| \geq 2$.

Consider the subgraph replacement operations in figure 6.16.

There are variants of γ_3 and γ_4 depending on the relative orientation of the subgraphs on the right and left. For instance γ_3 has $\{w_1, w_2, w_3, w_4\} = \{v_1, v_2, v_3, v_4\}$ but w_i need not necessarily be the same as v_i . The subgraph replacement operations $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, seem to provide a method of construction for graphs in $C_2(1,2)$ although no proof has been found. A conjecture is made.

Conjecture 6.16. If $G \in C_2(1,2; p, f)$, $f \geq 1$, G not a cycle, then G can be constructed from an element in $C_2(1,2; p-2, f)$ by $\gamma_1, \gamma_2, \gamma_3$, or γ_4 .

The statement of the above conjecture excludes the case $f = 0$. The graphs in $C_1(1,2; p, 0)$ correspond to rigid structures without substructures. If $G \in C_2(1,2; p, 0)$ then G has no vertices of degree two unless $p = 3$, in which case G is a 3-cycle. In fact $C_2(1,2; 3, 0) = K_3$, the complete graph on three vertices, but $C_2(1,2; 5, 0)$ and $C_2(1,2; 7, 0)$ are empty. However, $C_2(1,2; 9, 0) = K_{3,3}$, the complete bipartite graph on two sets of three vertices. The

development is again broken and $C_2(1,2; 11,0)$ is empty. $C_2(1,2; 13,0)$ is not empty (figure 6.17) and it seems that the conjecture might be extended by excluding these particular cases.

Figure 6.18 shows graphs in $C_2(1,2; p,1)$, $p \leq 10$, which indicates that the difficulties encountered in generating structures with small numbers of links do not arise for KCs with $f = 1$.

The corresponding result for graphs in $\bar{C}_2(1,2)$ to the conjecture 6.16 can be proved. The case for graphs in $\bar{C}_2(1,2; p,0)$ requires only the operation γ_4 and is given in Laman (1970, Theorem 6.4).

Proposition 6.17. If $G \in \bar{C}_2(1,2; p,f)$, G not a cycle, then G can be constructed from an element in $\bar{C}_2(1,2; p-2,f)$ by γ_1, γ_2 , or γ_4 .

Proof. If G has no degree three vertex in $T(G)$ then by proposition 6.15 γ_1 or γ_2 constructs G . If G has a degree three vertex in $T(G)$ adjacent to $\{v_1, v_2, v_3\}$, suppose there are $P(v_1, v_2)$, $P(v_2, v_3)$, $P(v_3, v_1) \subseteq V(G)$ containing $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_3, v_1\}$ respectively such that $f(P(v_1, v_2)) = f(P(v_2, v_3)) = f(P(v_3, v_1)) = 0$. Also suppose there are no subsets of these sets of vertices with the same properties. These sets are unique by using proposition 6.13.

However, there is a pair of vertices, say $\{v_1, v_2\}$ such that $v \in P(v_1, v_2)$. Thus G is constructed by γ_4 . If at least one of the sets $P(v_1, v_2)$, $P(v_2, v_3)$ or $P(v_3, v_1)$ does not exist then again γ_4 constructs G .

Proposition 6.18. If $G \in \bar{C}_2(2,2; p,f)$, G not a cycle, then G

can be constructed from an element in $\overline{C}_2(2,2; p-2, f)$ by γ_1, γ_2 , or γ_4 .

Proof. Similar to the proof of proposition 6.17 but using proposition 6.14 instead of proposition 6.15.

The difficulties in constructing graphs in $C_2(1,2; p, 0) = C_2(2,2; p, 0)$ are overcome for graphs in $\overline{C}_2(1,2; p, 0) = \overline{C}_2(2,2; p, 0)$ and figure 6.19 shows graphs in $\overline{C}_2(1,2; p, 0)$ for $p \leq 9$.

6.5 Planar KCs in $C_2(2,2)$ and $\overline{C}_2(2,2)$

Let $P_2(p, f)$ and $\overline{P}_2(p, f)$ denote planar graphs in $C_2(2,2; p, f)$ and $\overline{C}_2(2,2; p, f)$ respectively. If $G \in \overline{P}_2(p, f)$ has an embedding in the plane with t_i faces of degree $i \geq 3$, then

$$2f - 3 = \sum (i - 4)t_i.$$

Thus if $G \in \overline{P}_2(p, 0)$, then $t_3 \geq 2$ and, $P_2(p, 0)$, $p > 3$ is empty.

Also if $G \in P_2(p, 1)$ then $\sum (i - 4)t_i = 0$ and each embedding of G has all faces of degree four.

Consider an embedding of $G \in P_2(p, f)$ which has all finite faces degree four. The infinite face has degree $2f + 2$ and the embedding is a $[1/2(p - 3f - 1), 2f - 2]$ quadrangulation (Brown 1965), (figure 6.20(i)). Suppose that f diagonals or braces are added to the finite faces with at most one diagonal in each face such that an $[n, m]$ sub-quadrangulation has less than $1/2m + 1$ braces. The resulting graph is in $\overline{P}_2(p + f, 0)$ and is called a minimum bracing of the quadrangulation (figure 6.20(ii)).

An (a, b) square grid (figure 6.21(i)) is an embedding of an element in $P_2(a + b + 2ab, a + b - 1)$ with particular metric properties. It is interesting to examine how these metric

properties affect the placing of braces in order to ensure that the result is rigid. Such a bracing is minimum if it uses the smallest possible number of braces.

It has been shown (Bolker and Crapo 1977) that

(1) A bracing of an (a,b) square grid is minimum if and only if the braces correspond to the edges of a spanning tree in the complete bipartite graph $K_{a,b}$ whose edges represent the grid squares (figure 6.21(ii)).

As a consequence the following necessary condition on minimum bracings is given.

(2) Any permutation of rows and columns of an (a,b) square grid transforms one minimum bracing into another.

The necessary and sufficient condition for minimum bracings of a general quadrangular (a,b) grid corresponding to the (a,b) square grid, but without specific metric properties can be formulated as follows.

(3) A bracing of a quadrangular (a,b) grid is minimum if and only if there are $a + b - 1$ braced quadrangles such that no (a',b') , $a' \leq a$, $b' \leq b$ subgrid has more than $a' + b' - 1$ braced quadrangles.

A bracing of a quadrangular (a,b) grid which satisfies (3) together with the condition that any permutation of "rows" and "columns" of the quadrangular grid yields another bracing satisfying (3) is a minimum bracing of the corresponding (a,b) square grid. Thus the condition (2) seems to be the essential consequence of the particular metric properties of the square grid for placing the braces to ensure that the result is rigid.

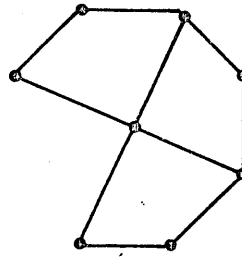


FIGURE 6.1 A graph in $C_1(2,2; 8,1)$ with a vertex of degree four.

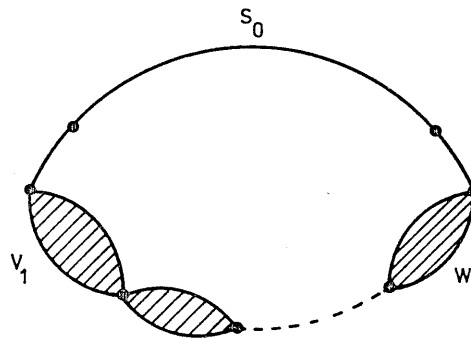


FIGURE 6.2 The suspended chain S_0 .

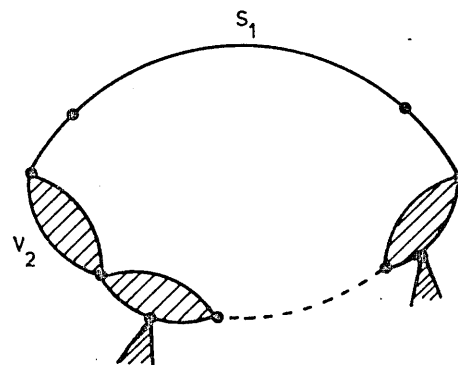
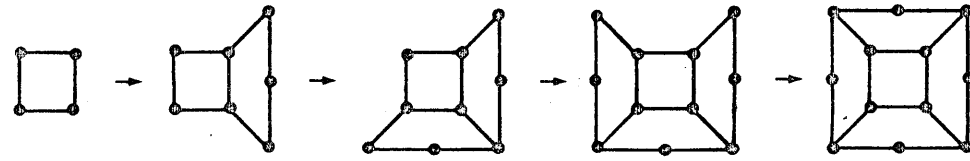
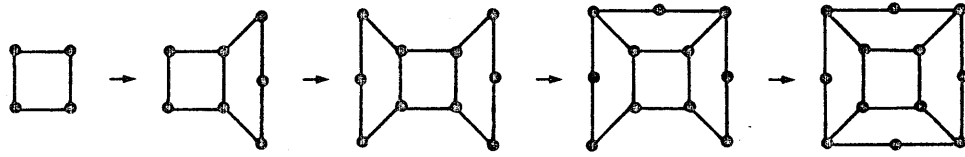


FIGURE 6.3 The block V_1 (figure 6.2).



(i)



(ii)

FIGURE 6.4 Two constructions of a graph in $C_1(2,2; 12,1)$ by suspended chain additions: (i) $a_1 = 2, a_3 = 2$ and (ii) $a_1 = 1, a_2 = 2, a_3 = 1$.

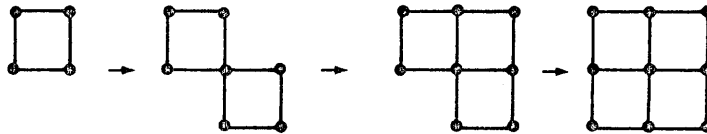
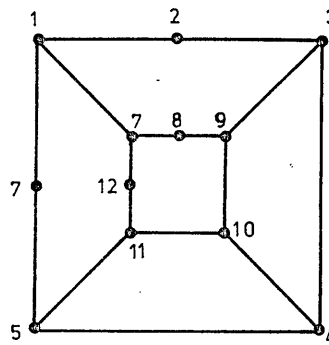


FIGURE 6.5 A construction of a graph in $C_1(2,2; 9,0)$ by suspended chain and cycle additions: $a_1 = 1, b_2 = 1, b_3 = 1$.



$P(3,4) = \{3,4\}$ $P(3,5) = \{3,4,5,9,10,11\}$
 $P(1,3) = \{1,2,3,4,5,6,7,8,9,10,11,12\}$
 $Q(3,4) = \{3,4,5,7,8,9,10,11,12\}$ or
 $\{1,2,3,4,5,6,9,10,11\}$
 $Q(3,5) = \{3,4,5\}$
 $Q(1,3) = \{1,2,3\}$ or $\{1,3,4,5,6,7,8,9,10,11,12\}$

FIGURE 6.6 The sets $P(u,v)$ and $Q(u,v)$

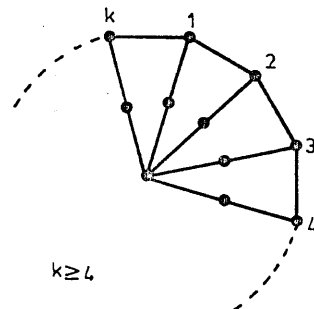
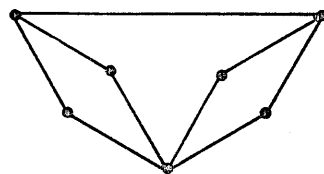


FIGURE 6.7 Graphs in $C_1(2,1)$ with all degree two vertices adjacent to a single vertex.

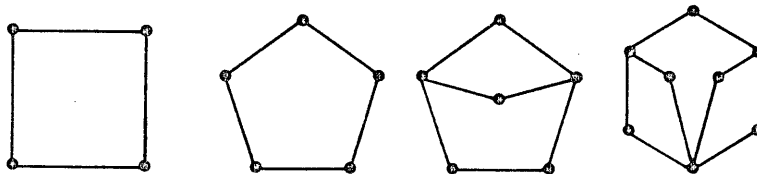


FIGURE 6.8 Graphs in $C_1(2,1)$, $f \geq 1$, with no remote degree two vertices.

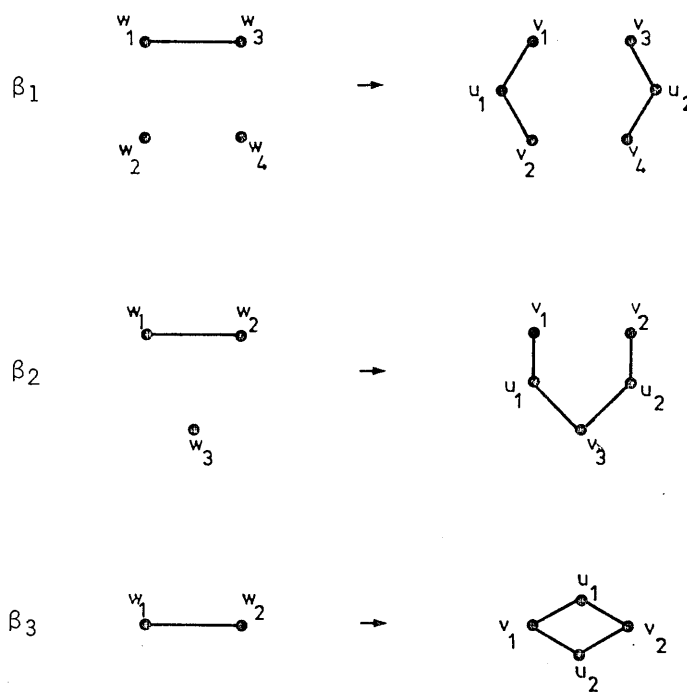


FIGURE 6.9 Operations β_1 , β_2 and β_3 .

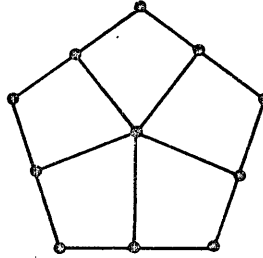


FIGURE 6.10 A graph in $C_1(2,1; 11,0)$ not constructible from a graph in $C_1(2,1; 9,0)$ by β_1 , β_2 or β_3 .

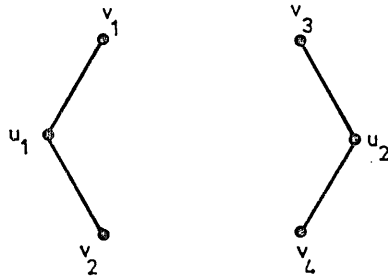
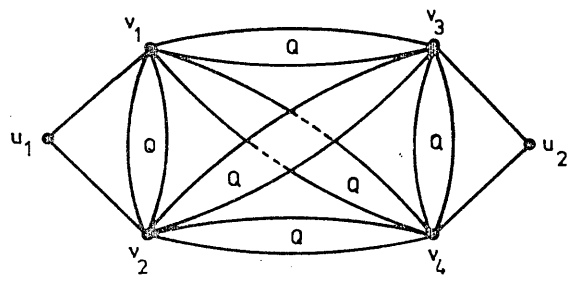
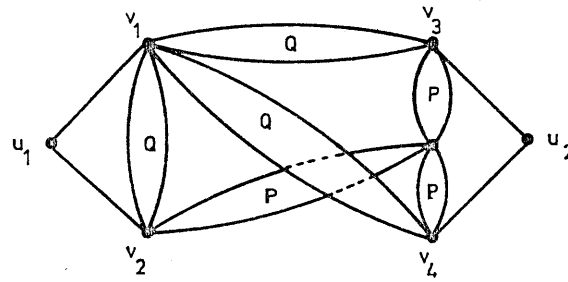


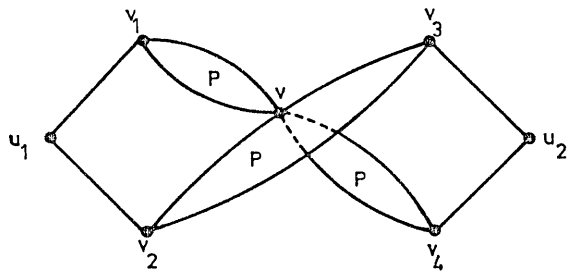
FIGURE 6.11 Configuration in G .



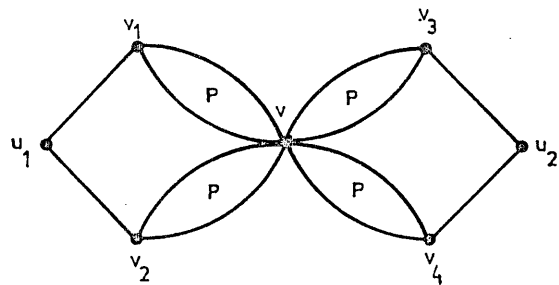
(a)



(b)



(c)



(d)

FIGURE 6.12 Four configurations for $R(u_1, u_2)$. P and Q denote subgraphs with $f = 1$ and $f = 2$ respectively.

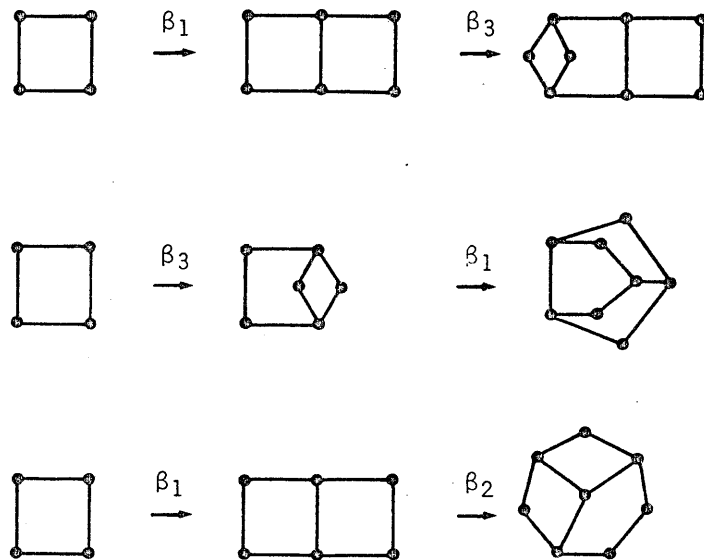


FIGURE 6.13 Construction of some graphs in $C_1(2,2; 8,1)$ by β_1 , β_2 and β_3 .

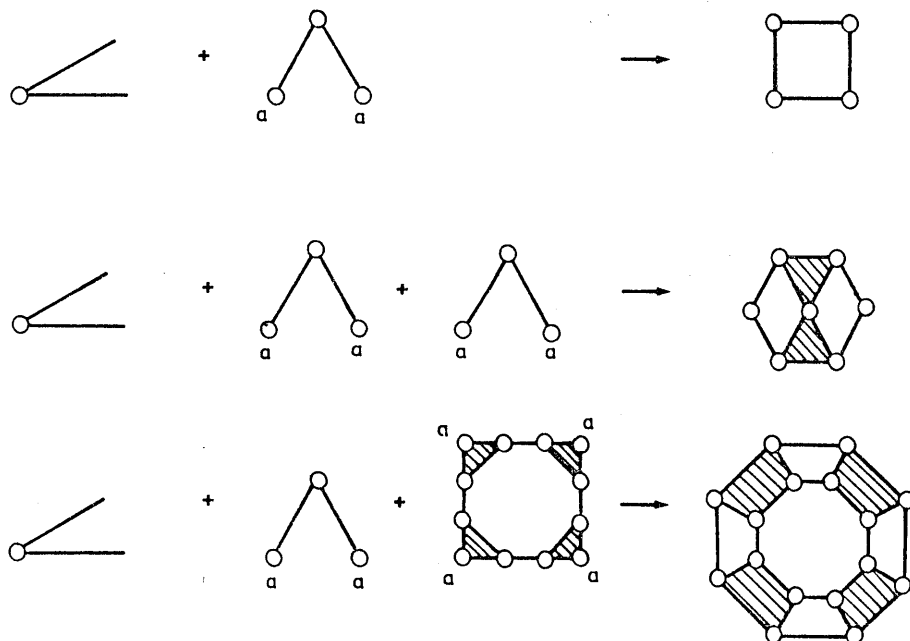


FIGURE 6.14 The Construction of coplanar KCs by Assur groups. The label "a" denotes a "joint of attachment".

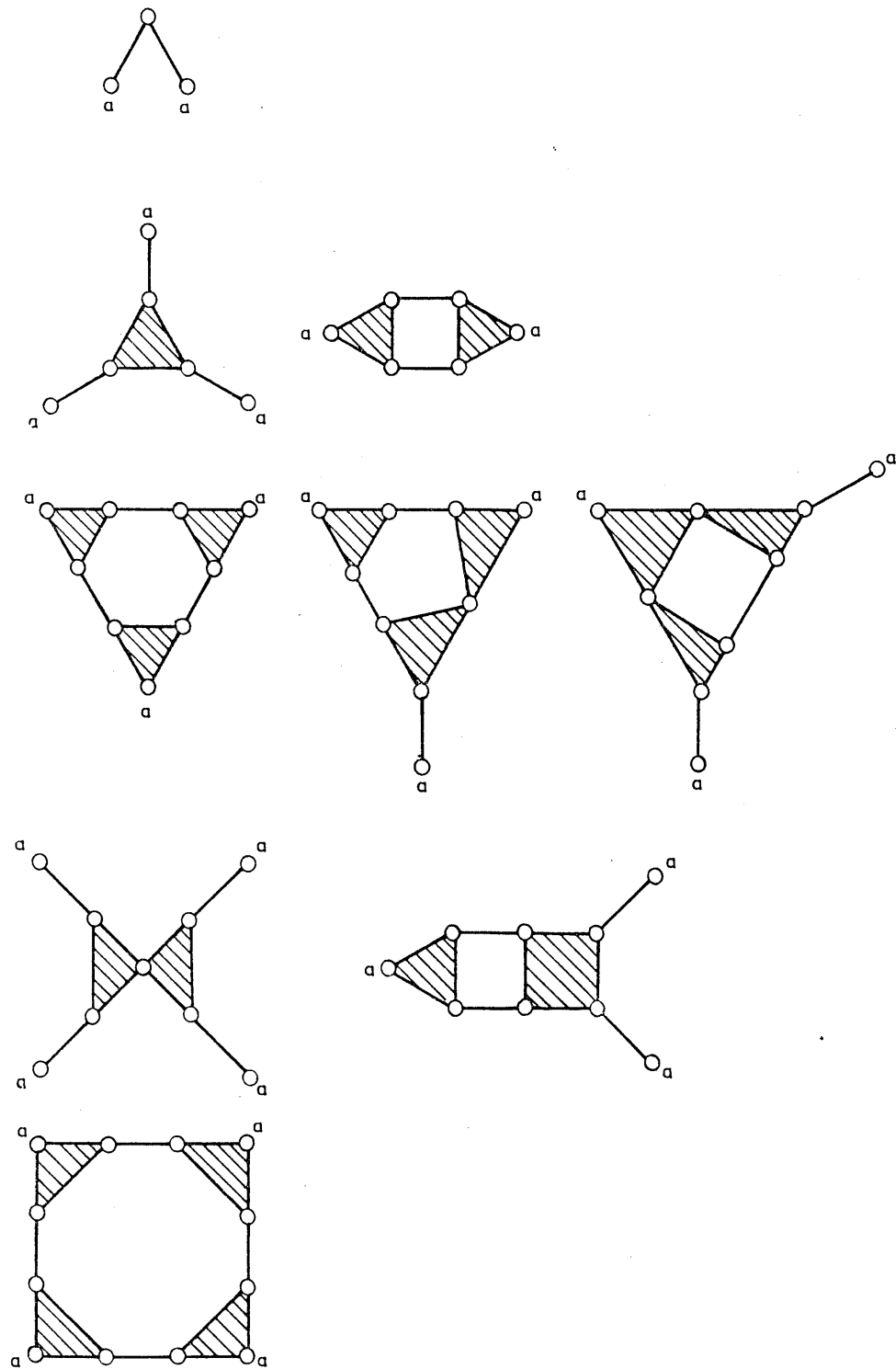


FIGURE 6.15 Some coplanar KCs required for Assur classification.

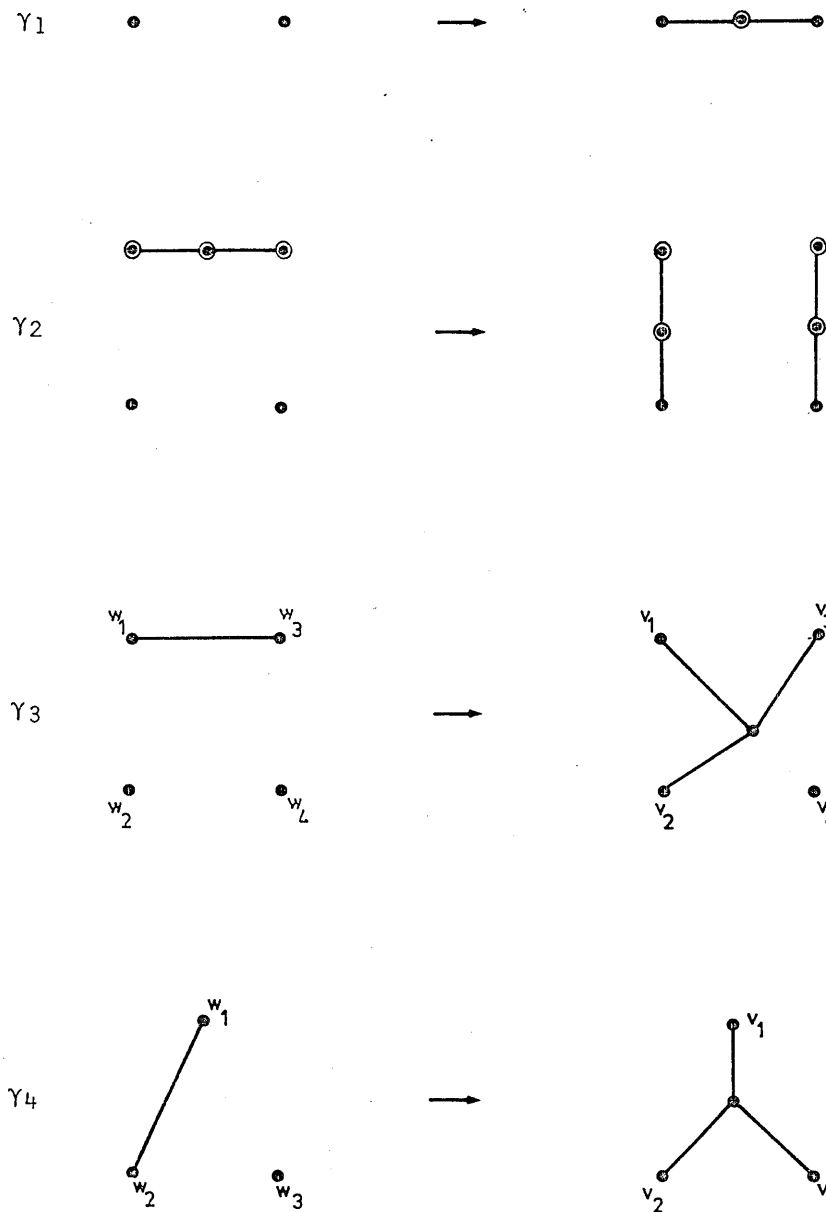
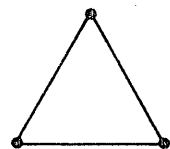


FIGURE 6.16 The operations γ_1 , γ_2 , γ_3 and γ_4 . The vertices marked \odot are degree two.



p

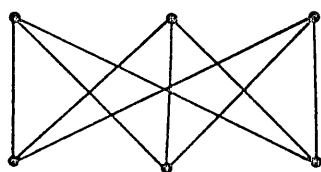
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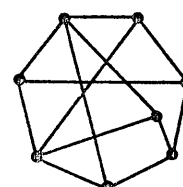
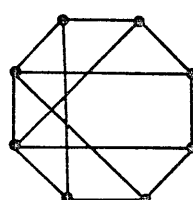
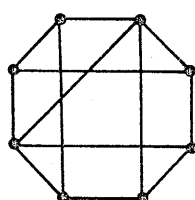
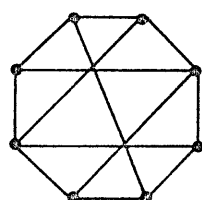
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9

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11



13

FIGURE 6.17 Graphs in $C_2(1,2; p,0)$, $p \leq 13$.

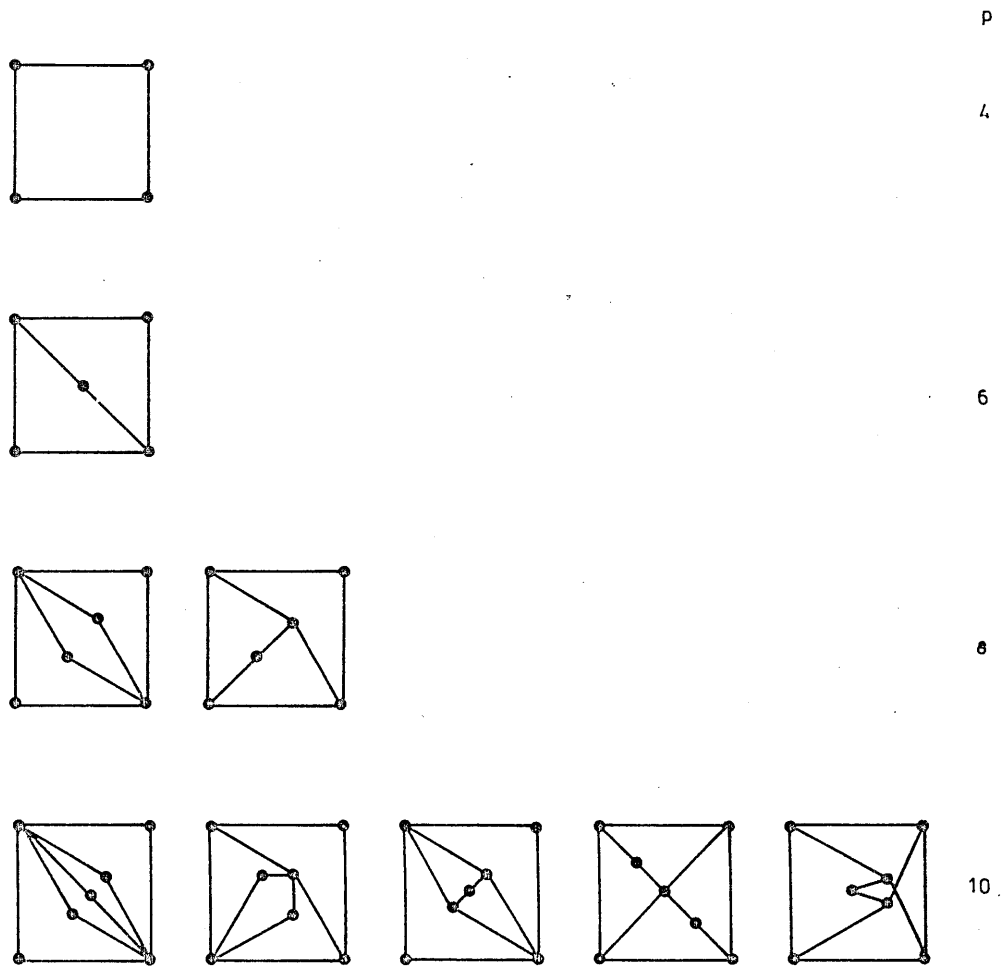


FIGURE 6.18 Graphs in $C_2(1,2; p,1)$, $p \leq 10$.

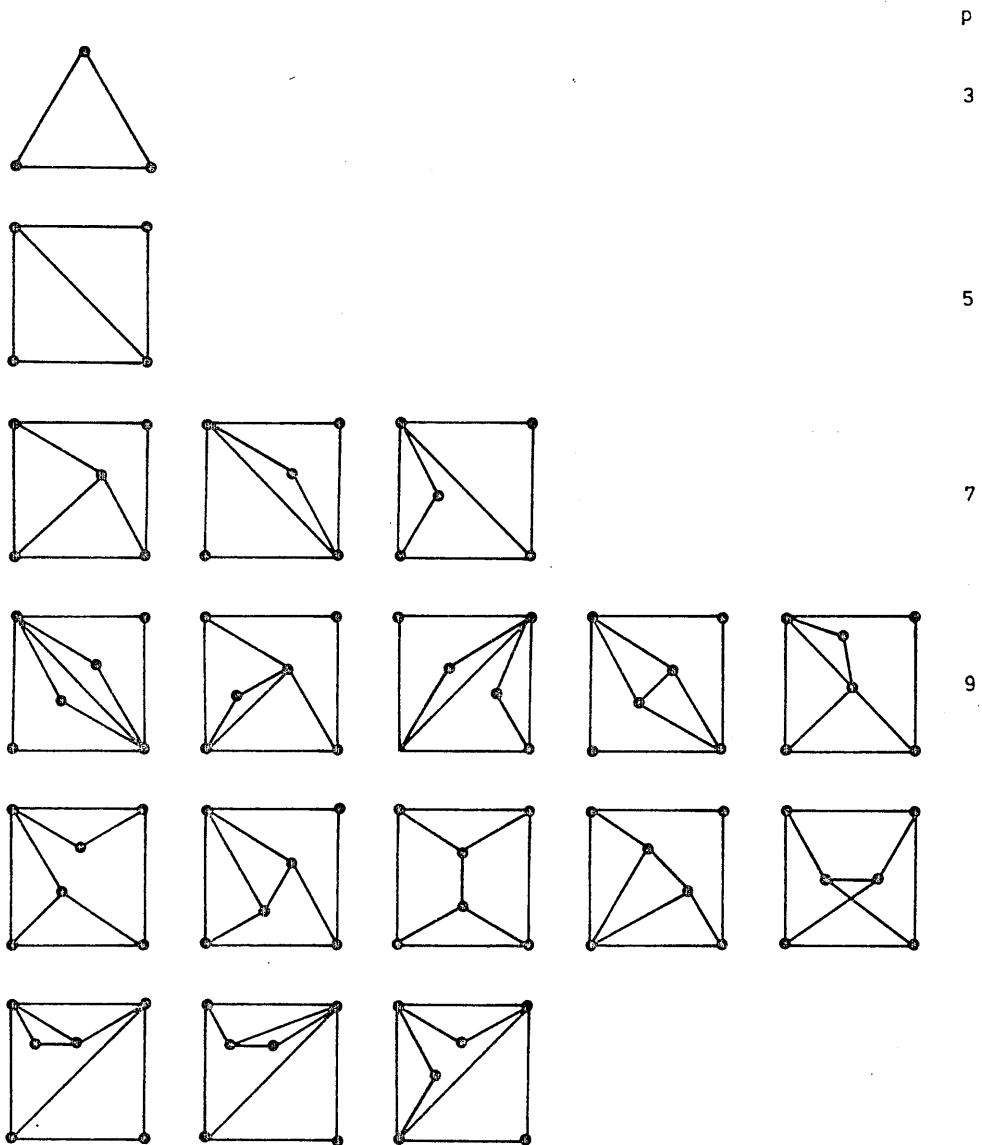


FIGURE 6.19 Graphs in $\overline{C}_2(1,2; p,0)$, $p \leq 9$.

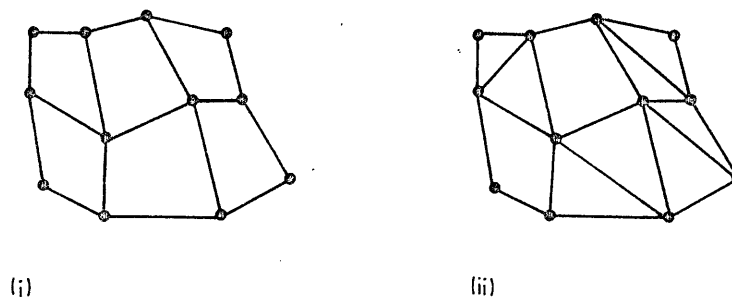


FIGURE 6.20 (i) A $[2,6]$ quadrangulation in $P_2(17,4)$ and (ii) a minimum bracing in $P_2(21,0)$.

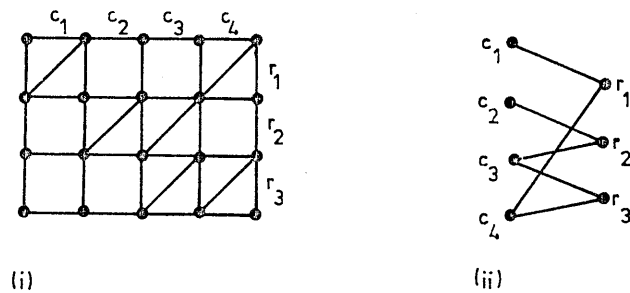


FIGURE 6.21 (i) A minimum bracing of a $(4,3)$ square grid and (ii) the corresponding spanning tree of $K_{4,3}$.

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